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# NEST SPACE DECOMPOSITION FOR AN OPERATOR IN $\mathscr{C}_\omega$

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**Abstract.** In this paper we prove that every operator X in the symmetrically normed ideal $\mathscr{C}_\omega$, $X \in B(H_1,H_2)$, can be decomposed as $X = X_1 + X_2$, where $X_1$ belongs to a nest space Op $\phi$ with $\phi$ to be a suitable (joint-continuous, zero preserving) map of sbspaces and $X_2$ belongs to a nest space Op $\sigma$, where $\sigma$ is the "adjoint" of the co-map of $\phi$.

## 1. Introduction and notations.

In [3] we study the convergence of the module triangular integral on symmetrically normed ideals and prove that given a continuous nest $\mathscr{E}$ every operator in the symmetrically normed ideal $\mathscr{C}_\omega$ is the sum of two operators; one in the considered nest algebra module $\mathscr{U}$ and one in the complementary module $\mathscr{U}^1$. In this paper we prove an analogous result, in a more general case, in accordance with the subspace maps and the corresponding linear spaces of operators. To be more specific we need the following definitions and notation which is taken from [1].

Let $H_1,H_2$ be two Hilbert spaces and $\mathscr{P}_1$ and $\mathscr{P}_2$ the sets of closed subspaces of $H_1$ and $H_2$ respectively. Let $\mathscr{M}(\mathscr{P}_1,\mathscr{P}_2)$ be the set of all joint-continuous, zero preserving maps from $\mathscr{P}_1$ to $\mathscr{P}_2$. Then for any $\phi \in \mathscr{M}(\mathscr{P}_1,\mathscr{P}_2)$, define the co-map $\psi:\mathscr{P}_1 \rightarrow \mathscr{P}_2$ of $\phi$ by $\psi(P_2) = \vee\{P_1 \in \mathscr{P}_1: \phi(P_1) \subseteq P_2\}$; $\psi$ is order preserving, meet continuous and satisfies $\psi(I) = I$. The range $\mathscr{S}_2$ of $\phi$ is clearly a joint-sub-semilattice of $\mathscr{P}_2$, that is complete with respect to joints. Also $0 \in \mathscr{S}_2$. The range $\mathscr{S}_1$ of $\psi$ is a meet-sub-semilattice of $\mathscr{P}_1$ that is complete with respect to meets. Also $I \in \mathscr{S}_1$. The semilattices $\mathscr{S}_1$ and $\mathscr{S}_2$ will be refered as the semilattices of $\phi$. To each map $\phi \in \mathscr{M}(\mathscr{P}_1,\mathscr{P}_2)$ there corresponds a set of operators, denoted by Op $\phi$, as follows

$$\text{Op}\,\phi = \{A \in \mathscr{B}(H_1,H_2): AP_1 \subseteq \phi(P_1) \text{ for all } P_1 \in \mathscr{P}_1\}.$$

When one of $\mathscr{S}_1$ and $\mathscr{S}_2$ is totally ordered then the corresponding set of operators Op $\phi$ is called **nest space.**

Now the result we prove here is the following. Every operator X in $\mathscr{C}_\omega$, $X \in \mathscr{B}(H_1,H_2)$ can be decomposed as $X = X_1 + X_2$ where $X_1 \in \mathrm{Op}\,\phi$ and $X_2 \in \mathrm{Op}\,\sigma$ and where $\phi$ is a map of subspaces so that the set Op $\phi$ is a nest space and $\sigma$ is the "adjoint" (see [1]) of the co-map of $\phi$. I wish to thank Dr. J.A. Erdos for bringing this problem to my attension and for all his valuable suggestions.

**2. The main result.**

Let $H_1,H_2$ be two Hilbert spaces and $H = H_2 \oplus H_1$. Consider a map $\phi \in \mathscr{M}(\mathscr{P}_1, \mathscr{P}_2)$, its semilattices $\mathscr{S}_1$ and $\mathscr{S}_2$ and the set $\mathscr{A}$ of operators in $\mathscr{B}(H_2 \oplus H_1)$ of the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \text{ with } A \in \mathrm{Alg}\,\mathscr{S}_2,\ B \in \mathrm{Alg}\,\mathscr{S}_1 \text{ and } X \in \mathscr{B}(H_1,H_2).$$

**Lemma 1.** If Op $\phi$ is a nest space then $\mathscr{A}$ is a nest algebra.

**Proof.** Since Op $\phi$ is a nest space the semilattices $\mathscr{S}_1$ and $\mathscr{S}_2$ are both totally ordered. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be the completions of $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively. We will prove that the nest of $\mathscr{A}$ is the set

$$\mathscr{E}\phi = \{N \oplus 0\colon N \in \mathscr{L}_2\} \cup \{H_2 \oplus M\colon M \in \mathscr{L}_1\}.$$

Clearly $\mathscr{E}\phi$ is a complete lattice containing 0 and I and it is easy to see that each member of $\mathscr{E}\phi$ is in Lat $\mathscr{A}$. Suppose now that $N \oplus M \in \mathrm{Lat}\,\mathscr{A}$. Then the inclusion

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} (N \oplus M) \subseteq N \oplus M \text{ for every } \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \in A \text{ implies}$$

that $AN+XM \subseteq N$ and $BM \subseteq M$ for any $A \in \mathrm{Lat}\,\mathscr{S}_2$, $B \in \mathrm{Alg}\,\mathscr{S}_1$ and $X \in \mathscr{B}(H_1,H_2)$. This is equivalent to $N \in \mathrm{Lat\,Alg}\,\mathscr{S}_2$ and $M = 0$ or $M \in \mathrm{Lat\,Alg}\,\mathscr{S}_1$ and $N = H_2$. Therefore $N \oplus M$ has one of the forms $M \oplus 0$ and $H_2 \oplus N$. Now since the lattices $\mathscr{L}_1$ and $\mathscr{L}_2$ are reflexive we have $N \in \mathscr{L}_2$ and $M \in \mathscr{L}_1$.

**Lemma 2.** Let $\mathscr{U} = \{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} : X \in \mathrm{Op}\,\phi\}$. Then $\mathscr{U}$ is a weakly closed $\mathscr{A}$-module.

**Proof.** It is enough to prove that $AY, YB \in \mathrm{Op}\,\phi$ for any $A \in \mathrm{Alg}\,\mathscr{S}_2$, $B \in \mathrm{Alg}\,\mathscr{S}_1$ and $Y \in \mathrm{Op}\,\phi$. But if $P_1 \in \mathscr{S}_1$ then

$$(AY)P_1 \subseteq A\phi(P_1) \subseteq \phi(P_1) \text{ and } (YB)P_1 \subseteq YP_1 \subseteq \phi(P_1).$$

Hence from the definition of Op $\phi$ we have AY, YB $\in$ Op $\phi$.

It is known that for a given nest algebra module $\mathscr{V}$ there exists a left continuous order homorphism from the corresponding nest $\mathscr{E}$ into itself which determines the module $\mathscr{V}$. In fact if $E \to \tilde{E}$, $E \in E$ is the order homomorphism then $\mathscr{V} = \{X \in \mathscr{B}(H): (I - \tilde{E})XE = 0 \text{ for all } E \in \mathscr{E}\}$ (see [2] or [3]). Also to this module $\mathscr{V}$ there corresponds the complementary module $\mathscr{U}^1 = \{X \in \mathscr{B}(H): \tilde{E}X(I - E) = 0 \text{ for all } E \in \mathscr{E}\}$.

**Lemma 3.** The left continuous order homomorphism which determines $\mathscr{U}$ is

$$N \oplus 0 \to (N \oplus 0)^{\sim} = 0 \text{ for any } N \in \mathscr{L}_2$$

$$H_2 \oplus M \to (H_2 \oplus M)^{\sim} = \phi(M) \oplus 0 \text{ for any } M \in \mathscr{L}_1$$

**Proof.** Let $M \in \mathscr{L}_1$. Then for any $Y \in$ Op $\phi$ we have

$$\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} (H_2 \oplus M) = Y(M) \oplus 0.$$

Therefore, from the reflexivity of $\mathscr{E}\phi$ and the definition of $\phi$ we have

$$(H_2 \oplus M)^{\sim} = \vee\{Y(M) \oplus 0 \text{ for every } Y \in \text{Op } \phi\} = \phi(M) \oplus 0.$$

In the sequel we consider the complementary module $\mathscr{U}^1$ of $\mathscr{U}$. An order homomorphism which determined $\mathscr{U}^1$ is the following

$$(N \oplus 0)^1 \to (H_2 \oplus H_1) \ominus (N \oplus 0)^{\sim} = H_2 \oplus H_1$$

$$(H_2 \oplus M)^1 \to (H_2 \oplus H_1) \ominus (\phi(M) \oplus 0) = \phi(M)^1 \oplus H_1.$$

We also consider the ideal $\mathscr{C}_\omega$ of compact operators X which is defined in terms of the eigenvalues of $(X^*X)^{1/2}$ (see [3], p. 308). Identifying the subspaces of $\mathscr{E}\phi$ with the orthogonal projections on them we denote the order homomorphism which determines the module $\mathscr{U}$ by $E \to \tilde{E}$, $E \in \mathscr{E}\phi$ and $E^- = \vee\{F \in \mathscr{E}\phi: F < E\}$ when $E \neq 0$ and $0^- = 0$. Now we state and prove the main result of this paper.

**Theorem 4.** Let $X \in \mathscr{C}_\omega$, $X \in \mathscr{B}(H_1,H_2)$ and $\phi$ be a subspace map. Then X can be decomposed as $X = X_1 + X_2$ where $X_1 \in$ Op $\phi$ and $X_2 \in$ Op $\sigma$ where $\sigma$ is the adjoint of the co-map of $\phi$.

**Proof.** Suppose $X \in \mathscr{C}_\omega$, $X \in \mathscr{B}(H_1,H_2)$ and hence

$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \in \mathscr{B}(H_1 \oplus H_2)$. Then from Theorem 24, [3] we have

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & X_2 \\ 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} 0 & X_1 \\ 0 & 0 \end{pmatrix} \in \mathscr{U}$ and $\begin{pmatrix} 0 & X_2 \\ 0 & 0 \end{pmatrix} \in \mathscr{U}^1$

Therefore $X_1 \in \mathrm{Op}\,\phi$ and what remains to be proved is to identify the module $\mathscr{U}^1$ with the $\mathrm{Op}\,\sigma$ where $\sigma$ is defined as $\sigma(M) = \psi^*(M) = [\Phi(M^1)]^1$ with $\psi$ to be the co-map of $\phi$. Thus $\mathrm{Op}\,\sigma = (\mathrm{Op}\,\psi)^*$. Let

$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$. Then $Z \in \mathscr{U}^1$ if and only if $\tilde{E}Z(I - E) = 0$

for every $E \in \mathscr{E}\phi$. Matricially this is equivalent to

$\begin{pmatrix} I_{\Phi(M)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{M^1} \end{pmatrix} = 0$ for every $M \in \mathscr{L}_1$.

A simple calculation shows that this is equivalent to $I_{\Phi(M)} Z_{12} I_{M^1} = 0$ for every $M \in \mathscr{L}_1$ and $Z_{11}$, $Z_{22}$, $Z_{21}$ arbitraries. Hence $Z_{12} I_{M^1} \subseteq \Phi(M)^1$ for every $M \in \mathscr{L}_1$. Therefore if $\sigma = \psi^*$ then $Z_{12} \in \mathrm{Op}\,\sigma$ and hence $X_2 \in \mathrm{Op}\,\sigma$. This completes the proof.

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