Perturbation of a nest algebra module

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1. *Introduction*

Fall, Arveson and Muhly (4) characterized the compact perturbation of nest algebras. In fact they proved that the compact perturbation of a nest algebra corresponding to a nest of projections is the algebra of operators which are quasitriangular relative to this nest. Erdos and Power (3) investigated weakly closed ideals and modules of nest algebras and these exhibit properties that are very close to the properties of the nest algebras themselves. They also showed that in certain cases, as in the case when the homomorphism which determines the nest algebra module is continuous, the results of Fall, Arveson and Muhly carry over to the more general situation. In this paper we provide a characterization of the compact perturbation of any nest algebra module.

The terms Hilbert space and projection will be used in this paper to mean separable complex Hilbert space and orthogonal projection. The set of all bounded linear operators from a Hilbert space H to a Hilbert space K will be denoted by $\mathscr{B}(H,K)$. When $H = K$ this set is simply denoted by $\mathscr{B}(H)$. By a module we mean a two-sided module. For a nest $\mathscr{E}$ of projections the corresponding nest algebra is denoted by $\operatorname{Alg}\mathscr{E}$. By $\mathscr{K}$ we denote the compact operators on H.

2. *Compact perturbations of a nest algebra module*

Let $\mathscr{E}$ and $\mathscr{F}$ be two order isomorphic nests acting on Hilbert spaces H and K respectively and let ϕ be the implementing isomorphism $\phi: \mathscr{E} \to \mathscr{F}$. The upper triangular operators with respect to the ordered pair $\{\mathscr{E}, \mathscr{F}\}$ (which in general do not form an algebra) are

$$J_\phi(\mathscr{E},\mathscr{F}) = \{X \in \mathscr{B}(H,K): (I-\phi(E))\,XE = 0,\, E \in \mathscr{E}\}.$$

It is proved in (5) that

$$d(X, J_\phi(\mathscr{E},\mathscr{F})) = \sup_{E\in\mathscr{E}} \|(I-\phi(E))\,XE\|, \quad X \in \mathscr{B}(H,K). \tag{1}$$

Here $d(\ ,\ )$ denotes the distance induced by the operator norm. It is mentioned in (3) that this result is also true when ϕ is an order homomorphism from a nest into itself. Since this is not obvious and the proof is not published and since we use it to prove our main result, we shall give a proof of that for the convenience of the reader.

LEMMA 1. *Let $\mathscr{E}$ be a nest of projections on a Hilbert space H and let $X \in \mathscr{B}(H)$. If $\phi: \mathscr{E} \to \mathscr{E}$ is an order homomorphism from $\mathscr{E}$ into itself and $\mathscr{A}(\mathscr{E},\phi) = \{X \in \mathscr{B}(H): (I-\phi(E))\,XE = 0,\, E \in \mathscr{E}\}$ then*

$$d(X, \mathscr{A}(\mathscr{E},\phi)) = \sup_{E\in\mathscr{E}} \|(I-\phi(E))\,XE\|. \tag{2}$$

(*Note that $\mathscr{A}(\mathscr{E},\phi)$ is an* $\operatorname{Alg}\mathscr{E}$*-module.*)

Proof. We shall prove (2) first for any finite subnest of $\mathscr{E}$. Let $\mathscr{E}_1 = \{E_1, E_2, \ldots, E_n\}$ be any finite subnest of $\mathscr{E}$ and let $\mathscr{F} = \{F_1, F_2, \ldots, F_k\}$, $k \leq n$, be the set of distinct images of the members of $\mathscr{E}_1$ under ϕ. Put $\mathscr{G} = \{G_j \in \mathscr{E}_1, 1 \leq j \leq k$, where G_j is the largest projection in $\mathscr{E}_1$ such that $\phi(G_j) = F_j\}$.

Then ϕ restricted to $\mathscr{G}$ is an order isomorphism from $\mathscr{G}$ onto $\mathscr{F}$. It is easy to see that the Alg $\mathscr{E}_1$-module $\mathscr{A}(\mathscr{E}_1, \phi)$ is equal to the set $J_\phi(\mathscr{G}, \mathscr{F})$ of upper triangular operators corresponding to the ordered pair $\{\mathscr{G}, \mathscr{F}\}$. Indeed, the inclusion $\mathscr{A}(\mathscr{E}_1, \phi) \subseteq J_\phi(\mathscr{G}, \mathscr{F})$ is obvious. Let $T \in J_\phi(\mathscr{G}, \mathscr{F})$ and an $E \in \mathscr{E}_1$. Let G_j be the projection in $\mathscr{G}$ such that $\phi(E) = \phi(G_j) = F_j$. From $(I - F_j)TG_j = 0$, since $E \leq G_j$ we have $(I - \phi(E))TE = 0$. This implies that $T \in \mathscr{A}(\mathscr{E}_1, \phi)$ and hence $\mathscr{A}(\mathscr{E}_1, \phi) \supseteq J_\phi(\mathscr{G}, \mathscr{F})$. Therefore

$$\mathscr{A}(\mathscr{E}_1, \phi) = J_\phi(\mathscr{G}, \mathscr{F})$$

as required.

Let $X \in \mathscr{B}(H)$. Then, by (1), we have

$$d(X, \mathscr{A}(\mathscr{E}_1, \phi)) = d(X, J_\phi(\mathscr{G}, \mathscr{F})) = \max_{1 \leq j \leq k} \|(I - F_j) X G_j\|.$$

But for any $E \in \mathscr{E}_1$ there exists a $G_j \in \mathscr{G}$ such that $E \leq G_j$ and $\phi(E) = \phi(G_j)$. Hence

$$\|(I - \phi(E)) X E\| = \|(I - \phi(E)) X G_j E\| \leq \|(I - F_j) X G_j\|.$$

Therefore

$$d(X, \mathscr{A}(\mathscr{E}_1, \phi)) = \max_{1 \leq i \leq n} \|(I - \phi(E_i)) X E_i\|.$$

Now the proof proceeds in the same way as in (1) and it is omitted.

Definition 2. Let $\mathscr{A}(\mathscr{E}, \phi) = \{X \in \mathscr{B}(H) : (I - \phi(E)) X E = 0$ for all $E \in \mathscr{E}\}$ be the Alg $\mathscr{E}$-module determined by the homomorphism $\phi: \mathscr{E} \to \mathscr{E}$. Define $Q\mathscr{A}(\mathscr{E}, \phi)$ to be the set of all X in $\mathscr{B}(H)$ such that:

(i) $(I - \phi(E)) X E \in \mathscr{K}$ for every $E \in \mathscr{E}$.

(ii) The set $\{(I - \phi(E)) X E, E \in \mathscr{E}\}$ is precompact in the norm topology.

The members of $Q\mathscr{A}(\mathscr{E}, \phi)$ are called the quasitriangular operators with respect to the Alg $\mathscr{E}$-module $\mathscr{A}(\mathscr{E}, \phi)$.

THEOREM 3. *If $\mathscr{A}(\mathscr{E}, \phi)$ is a nest algebra module determined by the homomorphism ϕ then*

$$Q\mathscr{A}(\mathscr{E}, \phi) = \mathscr{A}(\mathscr{E}, \phi) + \mathscr{K}.$$

Proof. We first prove that $Q\mathscr{A}(\mathscr{E}, \phi) \subseteq \mathscr{A}(\mathscr{E}, \phi) + \mathscr{K}$. The proof of this proceeds in a fashion analogous to that found in (4). Let $X \in Q\mathscr{A}(\mathscr{E}, \phi)$ and choose $\epsilon > 0$. We shall construct a compact operator K such that the distance from $X - K$ to $\mathscr{A}(\mathscr{E}, \phi)$ is at most ϵ. From theorem 1·1 of (4) and corollary 1·6 of (3), $\mathscr{A}(\mathscr{E}, \phi) + \mathscr{K}$ is normed closed. Hence since ϵ is arbitrary X will belong to $\mathscr{A}(\mathscr{E}, \phi) + \mathscr{K}$ and hence $Q\mathscr{A}(\mathscr{E}, \phi) \subseteq \mathscr{A}(\mathscr{E}, \phi) + \mathscr{K}$.

The set $\{(I - \phi(E)) X E, E \in \mathscr{E}\}$ is precompact and so it contains an ϵ-net

$$\{(I - \phi(E_j)) X E_j : 0 \leq j \leq n\}.$$

Hence for every $E \in \mathscr{E}$ there exists j such that $0 \leq j \leq n$ and

$$\|(I - \phi(E)) X E - (I - \phi(E_j)) X E_j\| < \epsilon.$$

Obviously we can assume that $0 = E_0 < E_1 < \ldots < E_n = I$. Put

$$P_j = E_j - E_{j-1}, \quad 0 \leq j \leq n,$$

and define

$$K = \sum_{j=1}^{n} (I-\phi(E_j))\, XP_j.$$

From condition (i) of definition 2 we have

$$(I-\phi(E_j))\, XP_j = (I-\phi(E_j))\, XE_j P_j \in \mathscr{K} \quad \text{for every } j\colon 0 \leqslant j \leqslant n,$$

and so K is a compact operator.

By Lemma 1, for any $T \in \mathscr{B}(H)$, $d(T, \mathscr{A}(\mathscr{E},\phi)) = \max_{E\in\mathscr{E}} \|(I-\phi(E))\, TE\|$; therefore to prove $d(X-K, \mathscr{A}(\mathscr{E},\phi)) \leqslant \epsilon$ it will be enough to prove that

$$\|(I-\phi(E))\,(X-K)\,E\| \leqslant \epsilon \quad \text{for every } E \in \mathscr{E}.$$

Fix $E\in\mathscr{E}$. Then there is $j\colon 0 \leqslant j \leqslant n$ so that $E_{j-1} \leqslant E \leqslant E_j$. We have

$$X-K = \sum_{k=1}^{n} XP_k - \sum_{k=1}^{n} (I-\phi(E_k))\, XP_k$$
$$= \sum_{k=1}^{n} \phi(E_k)\, XP_k$$

and therefore

$$(I-\phi(E))\,(X-K)\,E = \sum_{k=1}^{n} (I-\phi(E))\,\phi(E_k)\, XP_k E.$$

Now for $k > j$, $P_k E = 0$ and for $k < j$, since $E_k < E$ implies $\phi(E_k) \leqslant \phi(E)$, we have $(I-\phi(E))\,\phi(E_k) = 0$. Hence

$$(I-\phi(E))\,(X-K)\,E = (I-\phi(E))\,\phi(E_j)\, XP_j E$$
$$= \phi(E_j)\,(I-\phi(E))\, XEP_j.$$

Choose k, $0 \leqslant k \leqslant n$, such that

$$\|(I-\phi(E))\, XE - (I-\phi(E_k))\, XE_k\| \leqslant \epsilon.$$

Then

$$\begin{aligned}
&\|(I-\phi(E))\,(X-K)\,E\| \\
&\quad = \|\phi(E_j)\,(I-\phi(E))\, XEP_j\| \\
&\quad \leqslant \|\phi(E_j)\,(I-\phi(E))\, XEP_j - \phi(E_j)\,(I-\phi(E_k))\, XE_k P_j\| + \|\phi(E_j)\,(I-\phi(E_k))\, E_k P_j\| \\
&\quad \leqslant \epsilon + \|\phi(E_j)\,(I-\phi(E_k))\, XE_k P_j\|.
\end{aligned}$$

But $E_k P_j = 0$ for $k < j$ and for $k \geqslant j$, since $E_k \geqslant E_j$ implies

$$\phi(E_k) \geqslant \phi(E_j), \quad \phi(E_j)\,(I-\phi(E_k)) = 0.$$

Hence

$$\|\phi(E_j)\,(I-\phi(E_k))\, XE_k P_j\| = 0 \quad \text{and thus} \quad \|(I-\phi(E))\,(X-K)\,E\| \leqslant \epsilon.$$

For the reverse inclusion $\mathscr{A}(\mathscr{E},\phi) + \mathscr{K} \subseteq Q\mathscr{A}(\mathscr{E},\phi)$, let $X \in \mathscr{A}(\mathscr{E},\phi) + \mathscr{K}$. Then $X = A + K$ where $A \in \mathscr{A}(\mathscr{E},\phi)$ and $K \in \mathscr{K}$ and for any $E \in \mathscr{E}$

$$(I-\phi(E))\, XE = (I-\phi(E))\, AE + (I-\phi(E))\, KE = (I-\phi(E))\, KE \in \mathscr{K}.$$

Therefore X satisfies the first condition of the definition 2. To prove that X also satisfies condition (ii) it is sufficient to prove that the set $S = \{(I-\phi(E)\, KE\colon E \in \mathscr{E}\}$ is sequentially compact for any fixed compact operator K.

To show this, let $\{(I-\phi(E_n))\,KE_n\}_1^\infty$ be a sequence in S. We shall prove that this sequence has a convergent subsequence. Since $\mathscr{E}$ is a nest the sequence $\{E_n\}_1^\infty$ of projections in $\mathscr{E}$ contains either an increasing subsequence or a decreasing subsequence. Suppose that $\{E_n\}_1^\infty$ has an increasing subsequence. For the other case we work similarly. For convenience let $\{E_n\}_1^\infty$ be the increasing subsequence. Then, since $E_n \leqslant E_m$ implies $\phi(E_n) \leqslant \phi(E_m)$, the sequence $\{\phi(E_n)\}_1^\infty$ is also increasing. Then by corollary 2 of (2), they converge in the strong operator topology. Let E and F be their limits respectively. Since K is a compact operator, $\{KE_n\}_1^\infty$ converges to KE in norm and $\{(I-\phi(E_n))\,KE\}_1^\infty$ converges to $(I-F)\,KE$ in norm. Therefore

$$\begin{aligned}
&\|(I-\phi(E_n))\,KE_n-(I-F)\,KE\| \\
&\quad = \|(I-\phi(E_n))\,KE_n-(I-\phi(E_n))\,KE+(I-\phi(E_n))\,KE-(I-F)\,KE\| \\
&\quad \leqslant \|(I-\phi(E_n))\,(KE_n-KE)\|+\|(I-\phi(E_n))\,KE-(I-F)\,KE\| \\
&\quad \leqslant \|KE_n-KE\|+\|(I-\phi(E_n))\,KE-(I-F)\,KE\|,
\end{aligned}$$

which implies that $\{(I-\phi(E_n))\,KE_n\}_1^\infty$ converges to $(I-F)\,KE$ in norm. Hence S is sequentially compact and the proof is complete.

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