Triangular Integration with Respect to a Nest Algebra Module

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1. Introduction. The concept of the triangular integral of an operator on Hilbert space with respect to a complete nest was introduced by Brodskii [3] and in short time it was developed considerably [4], [5], [7]. Erdos and Longstaff in [5] studied the triangular integral with respect to a fixed nest and gave many applications of it. Erdos in [4] continues the study of the triangular integral by giving new proofs of some fundamental convergence theorems in this theory. In this paper we introduce the notion of the triangular integral with respect to a nest algebra-module and characterize the set of operators for which this integral is convergent. We also study the convergence of this integral on symmetrically-normed ideals and prove that, when the given nest $\mathscr{E}$ is continuous, every operator in the symmetrically-normed ideal $\mathscr{C}_\omega$ is the sum of two operators; one in the considered nest algebra-module $\mathscr{A}(\mathscr{E},\sim)$ and one in the complementary module $\mathscr{A}(\mathscr{E}^\perp,\sim)$. This paper owes much to the papers of Erdos and Longstaff [5] and Erdos [4].

2. Notation and preliminaries. Standard terminology and notation will be used (see, for example [5], [6]). The terms *Hilbert space*, *subspace* and *operator* will be used to mean *complex* Hilbert space, *closed* subspace and *bounded linear* operator on a Hilbert space respectively. The set of all operators on a Hilbert space H will be denoted by $\mathscr{B}(H)$. A set $\mathscr{E}$ of orthogonal projections is called a *nest of projections* if it is totally ordered by the usual ordering of operators. If a nest $\mathscr{E}$ contains 0 and I and is complete as a lattice then $\mathscr{E}$ is said to be a *complete* nest. For any member E of a complete nest $\mathscr{E}$, define

$$E^- = \bigvee \{F \in \mathscr{E} : F < E\}, \qquad \text{when } E \neq 0, \text{ and}$$

$$E^+ = \bigwedge \{F \in \mathscr{E} : F > E\}, \qquad \text{when } E \neq I.$$

By convention $0^- = 0$ and $I^+ = I$. If $E = E^-$ then $\mathscr{E}$ is called a *continuous* nest. Also if $E \to \tilde{E}$ is an order homomorphism of $\mathscr{E}$ into itself (that is $E \leq F$ implies $\tilde{E} \leq \tilde{F}$), define

$$E_* = \bigwedge \{\tilde{F} : F > E\} \qquad \text{(for the notation cf. [6]).}$$

The set of all operators which leave invariant the range of each member of a nest

$\mathscr{E}$ is called the *nest algebra* of $\mathscr{E}$ and is denoted by Alg $\mathscr{E}$. The terms *module* and *ideal* will be used to mean two-sided module and two-sided ideal. The rank one operator $x \to \langle x,e \rangle f$ will be denoted by $e \otimes f$.

Let $\mathscr{E}$ be a complete nest of projections. A *partition* $\mathscr{P}$ of $\mathscr{E}$ is a finite subset $\{E_i : 1 \leq i \leq n\}$ of $\mathscr{E}$ such that $0 = E_0 < E_1 < \ldots < E_n = I$. The projection $E_i - E_{i-1}$ will be denoted by ΔE_i. A partition $\mathscr{P}_1$ is a *refinement* of $\mathscr{P}$ if $\mathscr{P} \subseteq \mathscr{P}_1$. By the union of the partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of $\mathscr{E}$ we shall mean the partition $\mathscr{P}_1 \cup \mathscr{P}_2$ consisting of all points which belong to at least one of the partitions $\mathscr{P}_1$, $\mathscr{P}_2$. Clearly $\mathscr{P}_1 \subseteq \mathscr{P}_1 \cup \mathscr{P}_2$, $\mathscr{P}_2 \subseteq \mathscr{P}_1 \cup \mathscr{P}_2$. The partitions of $\mathscr{E}$ form a directed set under refinement. We recall, from [5], the definition of integration with respect to a nest $\mathscr{E}$. Let f be any function on $\mathscr{E}$ taking values in $\mathscr{B}(H)$. For a partition $\mathscr{P}$ of $\mathscr{E}$ let $\mathscr{F}_{\mathscr{P}} = \{F_i : 1 \leq i \leq n\}$ be a subset of $\mathscr{E}$ such that $E_{i-1} \leq F_i \leq E_i$, $i = 1, 2, \ldots, n$. Let

$$\psi(f,\mathscr{P},\mathscr{F}_{\mathscr{P}}) = \sum_{i=1}^{n} f(F_i)\, \Delta E_i.$$

We say that an operator X is the integral of f with respect to the nest $\mathscr{E}$ and we write

$$X = \int_{\mathscr{E}} f(E)dE$$

if, for arbitrary choice of $\mathscr{F}_{\mathscr{P}}$, $\psi(f,\mathscr{P},\mathscr{F}_{\mathscr{P}})$ converges in the norm topology of $\mathscr{B}(H)$ under refinement of $\mathscr{P}$, to the operator X.

3. The integration with respect to a nest algebra-module. In this section we give the definition of the triangular integral with respect to a nest algebra-module.

Let $\mathscr{E}$ be a complete nest of projections on a Hilbert space H, $\mathscr{A} = \text{Alg}\, \mathscr{E}$ be the corresponding nest algebra and $\mathscr{U}$ be a weakly closed $\mathscr{A}$-submodule of $\mathscr{B}(H)$ under operator multiplication. Let $\tilde{E}$ be the projection onto the closed linear span of $\{\text{range}(XE) : X \in \mathscr{U}\}$. Since $\mathscr{U}$ is a module, $\tilde{E}$ is invariant under $\mathscr{A}$ and so, by the reflexivity of complete nests, $\tilde{E} \in \mathscr{E}$. Obviously $\phi : E \to \tilde{E}$ is an order homomorphism from $\mathscr{E}$ into itself with $\tilde{0} = 0$ and moreover ϕ is left order continuous in the sense that

$$\lim_{E \uparrow F} \tilde{E} = [\lim_{E \uparrow F} E]^{\tilde{}} = \tilde{F}^{-} \qquad \text{(see [6])}.$$

If we put

$$\mathscr{A}(\mathscr{E},\sim) = \{X \in \mathscr{B}(H) : (I - \tilde{E})XE = 0 \text{ for all } E \in \mathscr{E}\}$$

then $\mathscr{A}(\mathscr{E},\sim)$ is also an $\mathscr{A}$-module and it is proved in [6] Theorem 1.5 that $\mathscr{A}(\mathscr{E},\sim) = \mathscr{U}$. This means that every weakly closed nest algebra module is always determined by a left order continuous homorphism ϕ with $\phi(0) = 0$. It is shown in [1] that this determination is unique. Therefore there is a one-to-one correspondence between weakly closed modules and left order continuous homomorphisms satisfying $\phi(0) = 0$.

In the sequel we shall always assume that every weakly closed module is determined by a left order continuous homomorphism $E \to \tilde{E}$ with $\tilde{0} = 0$ and we shall denote the given module by $\mathcal{A}(\mathcal{E},\sim)$.

Definition 1. Let f be a function on $\mathcal{E}$ taking values in $\mathcal{B}(H)$ and $\mathcal{U} = \mathcal{A}(\mathcal{E},\sim)$ be a weakly closed nest algebra module determined by the left order continuous homomorphism $\phi : E \to \tilde{E}$ with $\phi(0) = 0$. Consider the function $g = f \circ \phi$ on $\mathcal{E}$ which is the composition of ϕ and f.

The *module triangular integral* of f is defined to be the triangular integral of g with respect to the nest $\mathcal{E}$ and is denoted by

$$\int_{(\mathcal{E},\sim)} f(\tilde{E})dE = \int_{\mathcal{E}} g(E)dE.$$

For a fixed operator $A \in \mathcal{B}(H)$ let $f(E) = EA$. Then $g(E) = f(\phi(E)) = \tilde{E}A$ and the module triangular integral of f in that case is called the *module triangular integral of* A or the triangular integral of A with respect to the nest algebra module $\mathcal{A}(\mathcal{E},\sim)$ and is denoted by $\tilde{\mathcal{T}}(A)$.

If $\mathcal{P} = \{E_i : 1 \leq i \leq n\}$ is any partition of $\mathcal{E}$, $\Delta E_i = E_i - E_{i-1}$ and $\mathcal{F}_{\mathcal{P}} = \{F_i : 1 \leq i \leq n\}$ is a subset of $\mathcal{E}$ such that $E_{i-1} \leq F_i \leq E_i$, $i = 1, 2, \ldots, n$ then

$$\tilde{\mathcal{T}}(A) = \lim_{\mathcal{P}} \sum_{i=1}^{n} \tilde{F}_i A \Delta E_i.$$

When we choose $\mathcal{F}_{\mathcal{P}}$ to be $\{E_{i-1} : 1 \leq i \leq n\}$ and $\{E_i : 1 \leq i \leq n\}$ respectively then we denote the corresponding integrals

$$\tilde{\mathcal{L}}(A) = (m)\int_{\mathcal{E}} \tilde{E}AdE = \lim_{\mathcal{P}} \sum_{i=1}^{n} \tilde{E}_{i-1} A \Delta E_i$$

$$\tilde{\mathcal{U}}(A) = (M)\int_{\mathcal{E}} \tilde{E}AdE = \lim_{\mathcal{P}} \sum_{i=1}^{n} \tilde{E}_i A \Delta E_i.$$

Similarly

$$\tilde{\mathcal{D}}(A) = \int_{\mathcal{E}} d\tilde{E}AdE = \lim_{\mathcal{P}} \sum_{i=1}^{n} \Delta \tilde{E}_i A \Delta E_i.$$

For a partition $\mathcal{P}$ of $\mathcal{E}$ we shall write $\tilde{\mathcal{L}}_{\mathcal{P}}(A)$, $\tilde{\mathcal{U}}_{\mathcal{P}}(A)$ and $\tilde{\mathcal{D}}_{\mathcal{P}}(A)$ for the sums

$$\sum \tilde{E}_{i-1} A \Delta E_i, \qquad \sum \tilde{E}_i A \Delta E_i \quad \text{and} \quad \sum \Delta \tilde{E}_i A \Delta E_i$$

respectively.

Remark. Since every order homomorphism ϕ of $\mathcal{E}$ into $\mathcal{E}$ defines a weakly closed Alg $\mathcal{E}$-module, namely

$$\mathcal{A}(\mathcal{E},\phi) = \{X \in \mathcal{B}(H) : (I - \phi(E))XE = 0 \text{ for all } E \in \mathcal{E}\}$$

we can define a "module triangular integral" associated with the homomorphism ϕ, in the same way as above. Then we can consider the module $\mathcal{A}(\mathcal{E},\phi)$ and prove similar results as in the case of a left order continuous homomorphism ϕ satisfying $\phi(0) = 0$.

4. The convergence of the module triangular integral. In this section we characterize those operators for which the module triangular integral exists. We shall use for this the same techniques as in [5]. In [5] the main tool was the radical of the corresponding nest algebra as it was characterized by Ringrose [9]. In our case we shall define a subset of the nest algebra-module analogous to Ringrose's radical which will play the same role as the radical in [5].

Definition 2. We define the set $\mathcal{R}(\mathcal{E},\sim)$ to be the subset of $\mathcal{A}(\mathcal{E},\sim)$ such that, given $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $\mathcal{E}$ with the property

$$\|\Delta\tilde{E}_i X \Delta E_i\| < \varepsilon$$

for $1 \le i \le n$.

Lemma 3. *We have the following:*
(i) The set $\mathcal{R}(\mathcal{E},\sim)$ is norm closed.
(ii) $\mathcal{R}(\mathcal{E},\sim) = \{R \in \mathcal{A}(\mathcal{E},\sim) : \tilde{\mathcal{D}}(R) = 0\}$.

Proof. *(i):* Let $\{R_n\}_1^\infty$ be a sequence in $\mathcal{R}(\mathcal{E},\sim)$ which converges to the operator R in norm. Then given $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that

$$\|R_N - R\| < \frac{\varepsilon}{2} \qquad \text{for every } N > N(\varepsilon).$$

Take one such N. Then since $R_N \in \mathcal{R}(\mathcal{E},\sim)$, there exists a partition

$$\mathcal{P} = \{E_i\} \text{ of } \mathcal{E} \text{ with } \|\Delta\tilde{E}_i R_N \Delta E_i\| < \frac{\varepsilon}{2}, \qquad i = 1, 2, \ldots, n.$$

Therefore

$$\|\Delta\tilde{E}_i R \Delta E_i\| \le \|\Delta\tilde{E}_i(R - R_N)\Delta E_i\| + \|\Delta\tilde{E}_i R_N \Delta E_i\|$$

$$< \|R - R_N\| + \frac{\varepsilon}{2} < \varepsilon \qquad \text{for } i = 1, 2, \ldots, n$$

and hence $R \in \mathcal{R}(\mathcal{E},\sim)$. Equivalently $\mathcal{R}(\mathcal{E},\sim)$ is norm closed.

(ii): For any partition $\mathcal{P} = \{E_i\}$ of $\mathcal{E}$ and any $x \neq 0$, $x \in H$, we have

$$\|\tilde{\mathcal{D}}_{\mathcal{P}}(R)x\|^2 = \left\|\sum_{i=1}^n \Delta\tilde{E}_i R \Delta E_i x\right\|^2 = \sum_{i=1}^n \|\Delta\tilde{E}_i R \Delta E_i x\|^2$$

$$\le \sum_{i=1}^n (\|\Delta\tilde{E}_i R \Delta E_i\|^2 \|\Delta E_i x\|^2) \le [\max_{1\le i\le n} \|\Delta\tilde{E}_i R \Delta E_i\|]^2 \|x\|^2$$

and hence

$$(1) \qquad \|\tilde{\mathfrak{D}}_{\mathcal{P}}(R)\| \leq \max_{1\leq i\leq n}\|\Delta\tilde{E}_i R\Delta E_i\|.$$

Also for any $j : 1 \leq j \leq n$, $\Delta\tilde{E}_j R\Delta E_j = \Delta\tilde{E}_j\tilde{\mathfrak{D}}_{\mathcal{P}}(R)\Delta E_j$, thus

$$\|\Delta\tilde{E}_j R\Delta E_j\| = \|\Delta\tilde{E}_j\tilde{\mathfrak{D}}_{\mathcal{P}}(R)\Delta E_j\| \leq \|\tilde{\mathfrak{D}}_{\mathcal{P}}(R)\|, \qquad 1 \leq j \leq n.$$

Hence

$$(2) \qquad \max_{1\leq j\leq n}\|\Delta\tilde{E}_j R\Delta E_j\| \leq \|\tilde{\mathfrak{D}}_{\mathcal{P}}(R)\|.$$

Combining (1) and (2) we have

$$(3) \qquad \|\tilde{\mathfrak{D}}_{\mathcal{P}}(R)\| = \max_{1\leq i\leq n}\|\Delta\tilde{E}_i R\Delta E_i\|.$$

Since $\{\tilde{\mathfrak{D}}_{\mathcal{P}}\}$, as $\mathcal{P}$ varies, is a decreasing net of projections on $\mathcal{B}(H)$, it is clear from (3) that (ii) is equivalent to Definition 2 of the set $\mathcal{R}(\mathcal{E},\sim)$.

In the sequel we shall define a set of operators which is, in some sense, the "adjoint" of $\mathcal{A}(\mathcal{E},\sim)$ and a subset of this analogous to $\mathcal{R}(\mathcal{E},\sim)$. Both of them are needed for the study of the convergence of the module triangular integral.

Let ϕ be the order homomorphism from $\mathcal{E}^{\perp} = \{I - E \text{ for all } E \in \mathcal{E}\}$ into $\mathcal{E}^{\perp}$ such that $I - E \to I - \tilde{E}$. Then define

$$\begin{aligned}\mathcal{A}(\mathcal{E}^{\perp},\sim) &= \{X \in \mathcal{B}(H) : X(I - E) = (I - \tilde{E})X(I - E) \text{ for all } E \in \mathcal{E}\}\\ &= \{X \in \mathcal{B}(H) : \tilde{E}X(I - E) = 0 \text{ for all } E \in \mathcal{E}\}.\end{aligned}$$

It is easy to see that $\mathcal{A}(\mathcal{E}^{\perp},\sim)$ is an $(\text{Alg }\mathcal{E})^*$-module; for example if $A \in (\text{Alg }\mathcal{E})^*$ and $X \in \mathcal{A}(\mathcal{E}^{\perp},\sim)$ then $\tilde{E}XA(I - E) = \tilde{E}XEA(I - E) = 0$ for each $E \in \mathcal{E}$, and so $XA \in \mathcal{A}(\mathcal{E}^{\perp},\sim)$. Similarly $AX \in \mathcal{A}(\mathcal{E}^{\perp},\sim)$. We shall refer to $\mathcal{A}(\mathcal{E}^{\perp},\sim)$ as the *complementary* module of $\mathcal{A}(\mathcal{E},\sim)$.

When $\mathcal{E}$ is replaced by $\mathcal{E}^{\perp}$ we denote the corresponding triangular integrals, defined in Section 3, by $\tilde{\mathcal{T}}^{\perp}$, $\tilde{\mathcal{L}}^{\perp}$, $\tilde{\mathcal{U}}^{\perp}$ and $\tilde{\mathfrak{D}}^{\perp}$ and the corresponding sums by $\tilde{\mathcal{U}}^{\perp}_{\mathcal{P}}$, $\tilde{\mathcal{L}}^{\perp}_{\mathcal{P}}$ and $\tilde{\mathfrak{D}}^{\perp}_{\mathcal{P}}$ respectively. Obviously $\tilde{\mathcal{L}}^{\perp}_{\mathcal{P}}(A) = A - \tilde{\mathcal{U}}_{\mathcal{P}}(A)$ and $\tilde{\mathcal{U}}^{\perp}_{\mathcal{P}} = A - \tilde{\mathcal{L}}_{\mathcal{P}}(A)$ for any operator $A \in \mathcal{B}(H)$.

Definition 4. We define the set $\mathcal{R}(\mathcal{E}^{\perp},\sim)$ to be the subset of $\mathcal{A}(\mathcal{E}^{\perp},\sim)$ such that given any positive ε there exists a partition $\mathcal{P}$ of $\mathcal{E}$ with the property

$$\|\Delta\tilde{E}_i X\Delta E_i\| < \varepsilon \qquad \text{for } 1 \leq i \leq n.$$

Lemma 5. *Let* $\tilde{\mathcal{A}} = \mathcal{A}(\mathcal{E},\sim) \cap \mathcal{A}(\mathcal{E}^{\perp},\sim)$. *Then*

$$\tilde{\mathcal{A}} \cap \mathcal{R}(\mathcal{E},\sim) = \tilde{\mathcal{A}} \cap \mathcal{R}(\mathcal{E}^{\perp},\sim) = \mathcal{R}(\mathcal{E},\sim) \cap \mathcal{R}(\mathcal{E}^{\perp},\sim) = (0).$$

Proof. Let $X \in \tilde{\mathcal{A}}$. Then for any partition $\mathcal{P}$ of $\mathcal{E}$ we have $\Delta\tilde{E}_i X\Delta E_i = X\Delta E_i$ and hence $\tilde{\mathfrak{D}}_{\mathcal{P}}(X) = X$. Therefore $\tilde{\mathfrak{D}}(X) = X$. Hence from Lemma 3(ii) if $X \neq 0$, $X \notin \mathcal{R}(\mathcal{E},\sim)$ and so $\tilde{\mathcal{A}} \cap \mathcal{R}(\mathcal{E},\sim) = (0)$.

Similarly $\tilde{\mathcal{A}} \cap \mathcal{R}(\mathcal{E}^{\perp},\sim) = (0)$ and since $\mathcal{R}(\mathcal{E},\sim) \cap \mathcal{R}(\mathcal{E}^{\perp},\sim) \subseteq \tilde{\mathcal{A}}$ we have $\mathcal{R}(\mathcal{E},\sim) \cap \mathcal{R}(\mathcal{E}^{\perp},\sim) = [\mathcal{R}(\mathcal{E},\sim) \cap \mathcal{R}(\mathcal{E}^{\perp},\sim)] \cap \tilde{\mathcal{A}} = (0)$.

The following theorem describes the set of operators for which the integral $\tilde{\mathfrak{T}}$ exists.

Theorem 6. *The triangular integral of an operator A in $\mathcal{B}(H)$ with respect to the $\mathcal{A}$-module $\mathcal{A}(\mathcal{E},\sim)$ exists if and only if $A = R + S$ where $R \in \mathcal{R}(\mathcal{E},\sim)$ and $S \in \mathcal{R}(\mathcal{E}^{\perp},\sim)$. The operators R and S are then uniquely given by $R = \tilde{\mathfrak{T}}(A)$ and $S = \tilde{\mathfrak{T}}^{\perp}(A) = \lim_{\mathcal{P}} \tilde{\mathcal{L}}^{\perp}_{\mathcal{P}}(A)$.*

Proof. Suppose $A = R + S$. Then by Lemma 3(ii) and the definition of $\mathcal{R}(\mathcal{E}^{\perp},\sim)$, given any $\varepsilon > 0$ there exists a partition $\mathcal{P}_1$ of $\mathcal{E}$ for R and a partition $\mathcal{P}_2$ of $\mathcal{E}$ for S such that, if $\mathcal{P} = \{E_i, 1 \leq i \leq n\}$, $0 = E_0 < E_i < \ldots < E_n = I$ is any refinement of $\mathcal{P}_1 \cup \mathcal{P}_2$, then

$$\max_{1\leq i\leq n} \|\Delta \tilde{E}_i R \Delta E_i\| < \frac{\varepsilon}{3} \quad \text{and} \quad \max_{1\leq i\leq n} \|\Delta \tilde{E}_i S \Delta E_i\| < \frac{\varepsilon}{3}.$$

Now if $E_{i-1} \leq F_i \leq E_i$,

$$\sum_{i=1}^{n} \tilde{F}_i A \Delta E_i = \sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1})(R + S)\Delta E_i + \sum_{i=1}^{n} \tilde{E}_{i-1}(R + S)\Delta E_i.$$

It is easy to prove that if $R \in \mathcal{A}(\mathcal{E},\sim)$ then $\tilde{\mathcal{U}}_{\mathcal{P}}(R) = R$ and if $S \in \mathcal{A}(\mathcal{E}^{\perp},\sim)$ then $\tilde{\mathcal{L}}_{\mathcal{P}}(S) = 0$. Therefore

$$\begin{aligned}\left\|\sum_{i=1}^{n} \tilde{F}_i A \Delta E_i - R\right\| &= \left\|\sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1})(R + S)\Delta E_i - \sum_{i=1}^{n} \Delta \tilde{E}_i R \Delta E_i\right\| \\ &\leq \left\|\sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1})(R + S)\Delta E_i\right\| + \left\|\sum_{i=1}^{n} \Delta \tilde{E}_i R \Delta E_i\right\| \\ &\leq \max_{1\leq i\leq n} [\|(\tilde{F}_i - \tilde{E}_{i-1})(R + S)\Delta E_i\| + \|\Delta \tilde{E}_i R \Delta E_i\|] \\ &\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Hence $\tilde{\mathfrak{T}}(A)$ exists and $\tilde{\mathfrak{T}}(A) = R$.

Conversely if $\tilde{\mathfrak{T}}(A)$ exists then for a given $\varepsilon > 0$ there exists a partition $\mathcal{P} = \{E_i : 1 \leq i \leq n\}$ and for any choice of F_i with $E_{i-1} \leq F_i \leq E_i$ we have

$$\left\|\tilde{\mathfrak{T}}(A) - \sum_{i=1}^{n} \tilde{F}_i A \Delta E_i\right\| < \varepsilon. \tag{4}$$

But

$$A = \tilde{\mathcal{L}}_{\mathcal{P}}(A) + \tilde{\mathfrak{D}}_{\mathcal{P}}(A) + \tilde{\mathcal{L}}^{\perp}_{\mathcal{P}}(A). \tag{5}$$

Clearly $\tilde{\mathcal{L}}_{\mathcal{P}}(A) \in \mathcal{R}(\mathcal{E},\sim)$. Similarly $\tilde{\mathcal{L}}^{\perp}_{\mathcal{P}}(A) \in \mathcal{R}(\mathcal{E}^{\perp},\sim)$ since it is a "lower sum" of the type $\tilde{\mathcal{L}}_{\mathcal{P}}(A)$ but with respect to the complementary nest.

Now (4), which is equivalent to the existence of $\tilde{\mathfrak{T}}(A)$, implies that, as the partition is refined the first sum of (5) converges to $\tilde{\mathcal{T}}(A)$, the second sum converges to zero and hence the third sum is convergent. Let R and S be the norm limits of $\tilde{\mathcal{L}}_{\mathcal{P}}(A)$ and $\tilde{\mathcal{L}}_{\mathcal{P}}^{\perp}(A)$ respectively. Then since by Lemma 3(i) $\mathcal{R}(\mathcal{E},\sim)$ and $\mathcal{R}(\mathcal{E}^{\perp},\sim)$ are norm closed we have $R \in \mathcal{R}(\mathcal{E},\sim)$, $S \in \mathcal{R}(\mathcal{E}^{\perp},\sim)$ and hence $A = R + S$.

The uniqueness of R and S follows from Lemma 5. Indeed, if $A = R_1 + S_1 = R_2 + S_2$ then $R_1 - R_2 = S_2 - S_1 \in \mathcal{R}(\mathcal{E},\sim) \cap \mathcal{R}(\mathcal{E}^{\perp},\sim) = (0)$.

Corollary 7. *The set* $\mathcal{R}(\mathcal{E},\sim) = \{R \in \mathcal{A}(\mathcal{E},\sim) : \tilde{\mathcal{T}}(R)$ *exists*$\}$.

Proof. Since $\tilde{\mathcal{T}}(R)$ exists it follows $R = R_1 + S$ with $R_1 \in \mathcal{R}(\mathcal{E},\sim)$ and $S \in \mathcal{R}(\mathcal{E}^{\perp},\sim)$. If $R \in \mathcal{A}(\mathcal{E},\sim)$ then $S = R - R_1 \in \tilde{\mathcal{A}} \cap \mathcal{R}(\mathcal{E}^{\perp},\sim) = (0)$ and hence $R = R_1$.

We shall conclude this section by examining the convergence of the integrals $\tilde{\mathcal{L}}(A)$, $\tilde{\mathcal{U}}(A)$ and $\tilde{\mathcal{D}}(A)$ and establishing some relations between them. The following theorem gives necessary and sufficient conditions for the existence of those integrals.

Theorem 8. *If any two of $\tilde{\mathcal{L}}(A)$, $\tilde{\mathcal{U}}(A)$, $\tilde{\mathcal{D}}(A)$ exist, then so does the third. These integrals all exist if and only if $A = R + D + S$ with $R \in \mathcal{R}(\mathcal{E},\sim)$, $D \in \tilde{\mathcal{A}}$ and $S \in \mathcal{R}(\mathcal{E}^{\perp},\sim)$. The operators R, D and S are then uniquely given by*

$$R = \tilde{\mathcal{L}}(A)$$

$$D = \tilde{\mathcal{D}}(A) = \tilde{\mathcal{U}}(A) - \tilde{\mathcal{L}}(A)$$

$$S = A - \tilde{\mathcal{U}}(A).$$

Proof. For any partition $\mathcal{P}$ of $\mathcal{E}$ we have

$$\tilde{\mathcal{U}}_{\mathcal{P}}(A) = \tilde{\mathcal{D}}_{\mathcal{P}}(A) + \tilde{\mathcal{L}}_{\mathcal{P}}(A).$$

Hence it is clear that if two of $\tilde{\mathcal{L}}(A)$, $\tilde{\mathcal{U}}(A)$ and $\tilde{\mathcal{D}}(A)$ exist then so does the third. Also since

$$A = \tilde{\mathcal{L}}_{\mathcal{P}}(A) + \tilde{\mathcal{D}}_{\mathcal{P}}(A) + [A - \tilde{\mathcal{U}}_{\mathcal{P}}(A)]$$

if all the integrals exist we shall have

$$A = \tilde{\mathcal{L}}(A) + \tilde{\mathcal{D}}(A) + [A - \tilde{\mathcal{U}}(A)].$$

Using the fact that $\tilde{\mathcal{L}}(A) \in \mathcal{R}(\mathcal{E},\sim)$ and $A - \tilde{\mathcal{U}}(A) \in \mathcal{R}(\mathcal{E}^{\perp},\sim)$, it remains to prove that $\tilde{\mathcal{D}}(A) \in \tilde{\mathcal{A}}$. For this let $E \in \mathcal{E}$ and take a partition $\mathcal{P} = \{E_i : 1 \le i \le n\}$ of $\mathcal{E}$ such that E is a member of $\mathcal{P}$. Then if $E = E_j$ for some $j : 1 \le j \le n$, we have

$$(6)\quad \tilde{E}\tilde{\mathcal{D}}_{\mathcal{P}}(A) = \tilde{E}_j \sum_{i=1}^{n} \Delta\tilde{E}_i A \Delta E_i = \sum_{i=1}^{j} \Delta\tilde{E}_i A \Delta E_i = \sum_{i=1}^{n} \Delta\tilde{E}_i A \Delta E_i E_j = \tilde{\mathcal{D}}_{\mathcal{P}}(A) E_j$$

and it is clear that (6) holds for any partition of $\mathcal{E}$ which is a refinement of $\mathcal{P}$.

Hence $\tilde{E}\tilde{\mathfrak{D}}(A) = \tilde{\mathfrak{D}}(A)E$ and since E is arbitrary, this is true for all $E \in \mathscr{E}$. That is, $\tilde{\mathfrak{D}}(A) \in \tilde{\mathscr{A}}$.

Conversely, if $A = R + D + S$ with $R \in \mathscr{R}(\mathscr{E},\sim)$, $S \in \mathscr{R}(\mathscr{E}^{\perp},\sim)$ and $D \in \tilde{\mathscr{A}}$ then, as in the proof of Theorem 6, we have

$$\tilde{\mathscr{L}}_{\mathscr{P}}(D) = \tilde{\mathscr{L}}_{\mathscr{P}}(S) = 0$$

for all partitions of $\mathscr{E}$ and so

$$\tilde{\mathscr{L}}_{\mathscr{P}}(A) = \tilde{\mathscr{L}}_{\mathscr{P}}(R).$$

Also $\tilde{\mathscr{U}}_{\mathscr{P}}(R) = R$. From Lemma 3(ii), given $\varepsilon > 0$ there exists a partition $\mathscr{P}$ of $\mathscr{E}$ such that

$$\|R - \tilde{\mathscr{L}}_{\mathscr{P}}(R)\| = \|\tilde{\mathfrak{D}}_{\mathscr{P}}(R)\| < \varepsilon.$$

Hence $\tilde{\mathscr{L}}(A)$ exists and is equal to R. Also since $D \in \tilde{\mathscr{A}}$, working as in Lemma 5, we get $\tilde{\mathfrak{D}}(D) = D$. Moreover since $\tilde{\mathfrak{T}}(R)$, $\tilde{\mathfrak{T}}(S)$ exist we have $\tilde{\mathfrak{D}}(R) = \tilde{\mathfrak{D}}(S) = 0$. Hence $\tilde{\mathfrak{D}}(A) = \tilde{\mathfrak{D}}(D) = D$. Now the existence of $\tilde{\mathscr{L}}(A)$ and $\tilde{\mathfrak{D}}(A)$ and the first part of the theorem imply that $\tilde{\mathscr{U}}(A)$ also exists and

$$\tilde{\mathscr{U}}(A) = \tilde{\mathfrak{D}}(A) + \tilde{\mathscr{L}}(A) = D + R = A - S$$

from which we get

$$S = A - \tilde{\mathscr{U}}(A).$$

Corollary 9. *If $A \in \mathscr{A}(\mathscr{E},\sim)$ then*

(i) $\tilde{\mathscr{L}}(A)$ exists if and only if $\tilde{\mathfrak{D}}(A)$ exists and in this case $A = \tilde{\mathfrak{D}}(A) + \tilde{\mathscr{L}}(A)$.

(ii) $\tilde{\mathfrak{T}}(A)$ exists if and only if $\tilde{\mathfrak{D}}(A) = 0$ and in this case $A = \tilde{\mathfrak{T}}(A)$.

Proof. Since for $A \in \mathscr{A}(\mathscr{E},\sim)$ we have $\tilde{\mathscr{U}}(A) = A$, Theorem 8 implies (i). Condition (ii) is immediate from Lemma 3(ii) and Corollary 7.

Next we consider the module triangular integral of a compact operator. We shall use the following which is Lemma 4.1 in [5].

Lemma 10. *Let K be a compact operator and let $\mathscr{F}$ be a totally ordered family of projections. Then*

$$\lim_{E\in\mathscr{F}} \|FK - EK\| = 0$$

where $F = \sup \mathscr{F}$ and the limit is taken as E increases.

Let $\mathscr{E}$ be a complete nest and $E \to \tilde{E}$ be a left order continuous homomorphism of $\mathscr{E}$ into $\mathscr{E}$. Recall the definition of G_* for a $G \in \mathscr{E}$. That is, $G_* = \inf\{\tilde{E} : E > G\}$. Then we have the following

Lemma 11. *Let $\mathscr{E}$ be as above and K be a compact operator. Then*

(i) $\lim_{E<G} \|\tilde{G}^{-}KG^{-} - \tilde{E}KE\| = 0$, $G \in \mathscr{E}$ and the limit is taken as E increases.

(ii) $\lim_{E>G} \|G_{*}KG^{+} - \tilde{E}KE\| = 0$, $G \in \mathscr{E}$ and the limit is taken as E decreases.

Proof. (*i*) Since $E \to \tilde{E}$ is left order continuous we have $\lim_{E<G} \tilde{E} = \tilde{G}^-$. Also

$$\|\tilde{G}^-KG^- - \tilde{E}KE\| = \|\tilde{G}^-KG^- - \tilde{E}KG^- + \tilde{E}KG^- - \tilde{E}KE\|$$
$$\leq \|\tilde{G}^-K - \tilde{E}K\| + \|KG^- - KE\|.$$

From this inequality and the previous lemma we get (i). The proof of (ii) is similar to (i) and is omitted.

In the sequel we prove that $\tilde{\mathfrak{D}}(K)$ exists for every compact operator K and give an expression of $\tilde{\mathfrak{D}}(K)$ in terms of projections.

Proposition 12. *Let K be a compact operator, $\mathcal{A}(\mathcal{E},\sim)$ be a weakly closed module and $\mathcal{F} = \{F = E - E^-$ for all $E \in \mathcal{E}, E \neq E^-\}$. Then, if $\tilde{F} = \tilde{E} - \tilde{E}^-$, we have*

$$\tilde{\mathfrak{D}}(K) = \sum_{F\in\mathcal{F}} \tilde{F}KF.$$

Proof. The proof of this proposition is modelled on the proof of Lemma 4.3 of [5]. Put

$$K_0 = K - \sum_{F\in\mathcal{F}} \tilde{F}KF.$$

For any $G \in \mathcal{E}$ we have $(\tilde{G} - \tilde{G}^-)K_0(G - G^-) = 0$. From the definition of G^- and Lemma 11 we have that, for a given $\varepsilon > 0$ there exists a projection $E_G \in \mathcal{E}$ such that $E_G < G$ and

$$\|(\tilde{G} - \tilde{E}_G)K_0(G - E_G)\| < \varepsilon.$$

Similarly from the definition of G_* (note that $(\tilde{G}^+ - \tilde{G})K_0(G^+ - G) = 0$ implies $(G_* - \tilde{G})K_0(G^+ - G) = 0$) and Lemma 11(ii), there exists a projection $F_G \in \mathcal{E}$ such that $F_G > G$ and

$$\|(\tilde{F}_G - \tilde{G})K_0(F_G - G)\| < \varepsilon.$$

Therefore for each projection $E \in \mathcal{E}$, belonging in the order interval (E_G, F_G) we have

$$\|(\tilde{E} - \tilde{G})K_0(E - G)\| < \varepsilon. \tag{6'}$$

As G varies over $\mathcal{E}$ the set of order intervals $\{(E_G,F_G)\}$ is an open cover of the compact space $\mathcal{E}$ and so it has a finite subcover $\{(E_{G_i},F_{G_i}) : 1 \leq i \leq n\}$. If $\mathcal{P}$ is a partition of $\mathcal{E}$ containing each of the projections E_{G_i}, G_i, F_{G_i}, $i = 1, 2, \ldots, n$ then since

$$\|\tilde{\mathfrak{D}}_{\mathcal{P}}(K_0)\| \leq \max_{E_j\in\mathcal{P}} \|\Delta\tilde{E}_j K_0 \Delta E_j\| \qquad \text{(Proof as in Lemma 3)},$$

(6′) implies $\|\tilde{\mathfrak{D}}_{\mathcal{P}}(K_0)\| < \varepsilon$. Also

$$\tilde{\mathfrak{D}}_{\mathcal{P}}(K_0) = \tilde{\mathfrak{D}}_{\mathcal{P}}(K) - \sum_{F\in\mathcal{F}} \tilde{F}KF.$$

Therefore $\tilde{\mathfrak{D}}(K) = \sum_{F\in\mathcal{F}} \tilde{F}KF$ and the proof is completed.

Corollary 13. *Let K be a compact operator in* $\mathcal{A}(\mathcal{E},\sim)$. *Then* $K = K_1 + K_0$ *where* $K_1 = \tilde{\mathfrak{D}}(K)$ *and* $K_0 = \tilde{\mathfrak{L}}(K)$.

Proof. By the previous proposition $\tilde{\mathfrak{D}}(K)$ exists. Hence by Corollary 9(i) $\tilde{\mathfrak{L}}(K)$ also exists and $K = \tilde{\mathfrak{D}}(K) + \tilde{\mathfrak{L}}(K)$.

Proposition 14. *Let K be a compact operator in* $\mathcal{A}(\mathcal{E},\sim)$. *Then*

(i) The integral $\tilde{\mathfrak{I}}(K)$ *exists if* $\sum (\tilde{E} - \tilde{E}^-)K(E - E^-) = 0$ *or if* $\mathcal{E}$ *is a continuous nest, and in both cases we have* $\tilde{\mathfrak{I}}(K) = K$.

(ii) If the nest $\mathcal{E}$ *is continuous and the order homomorphism* $E \rightarrow \tilde{E}$ *is such that* $\tilde{E} \leq E$ *for all* $E \in \mathcal{E}$, *the operator K is uniquely determined by its real or imaginary part.*

Proof. Proposition 12 implies $\tilde{\mathfrak{D}}(K) = 0$ in both cases and hence part (i) follows from Corollary 9(ii). For part (ii), when $E \leq \tilde{E}$ for all $E \in \mathcal{E}$ then $\mathcal{A}(\mathcal{E},\sim)$ is an ideal of Alg $\mathcal{E}$ and $K^* \in \mathcal{A}(\mathcal{E}^\perp,\sim)$. Since $\mathcal{E}$ is continuous, $\tilde{\mathfrak{D}}^\perp(K^*) = \tilde{\mathfrak{D}}(K^*) = 0$. Hence $\tilde{\mathfrak{I}}^\perp(K^*)$ exists and $\tilde{\mathfrak{I}}^\perp(K^*) = K^*$. Therefore, if $K = X + iY$ is the decomposition of K into real and imaginary parts, then $K = \tilde{\mathfrak{I}}(K) = 2\tilde{\mathfrak{I}}(X) = 2i\tilde{\mathfrak{I}}(Y)$.

5. The module triangular integral on symmetrically-normed ideals. In this section we prove that the nets $\{\tilde{\mathfrak{L}}_{\mathcal{F}}\}$, $\{\tilde{\mathfrak{U}}_{\mathcal{F}}\}$, $\{\tilde{\mathfrak{D}}_{\mathcal{F}}\}$ converge on the C_p, $1 \leq p \leq \infty$ classes and for any $A \in C_\omega$, the nets $\{\tilde{\mathfrak{L}}_{\mathcal{F}}(A)\}$, $\{\tilde{\mathfrak{U}}_{\mathcal{F}}(A)\}$, $\{\tilde{\mathfrak{D}}_{\mathcal{F}}(A)\}$ converge in $\mathcal{B}(H)$. We also prove that, when $\mathcal{E}$ is continuous, every operator $X \in \mathcal{C}_\omega$ is written as $X = X_1 + X_2$ where $X_1 \in \mathcal{A}(\mathcal{E},\sim)$ and $X_2 \in \mathcal{A}(\mathcal{E}^\perp,\sim)$, and the condition $X \in \mathcal{C}_\omega$ is necessary and sufficient for the module triangular integral $\tilde{\mathfrak{I}}(X)$ to exist for every continuous nest $\mathcal{E}$ and any $\mathcal{A}(\mathcal{E},\sim)$.

We recall now from [4] the definitions of $\mathcal{C}_p$, $1 \leq p < \infty$ classes and of $\mathcal{C}_\omega$ and, its adjoint, $\mathcal{C}_\Omega$ symmetrically-normed ideals. (For more details see [7] or [8].) The characteristic numbers $(s_i(A))$ of the compact operator A are the sequence of eigenvalues of $(A^*A)^{1/2}$ in decreasing order and repeated according to multiplicity. Then

$$\mathcal{C}_p = \left\{A : \sum_{i=1}^{\infty} s_i(A)^p < \infty\right\}$$

$$\mathcal{C}_\omega = \left\{A : \sum_{i=1}^{\infty} \frac{1}{i} s_i(A) < \infty\right\}$$

$$\mathcal{C}_\Omega = \left\{A : \sup_n \frac{\sum_{i=1}^{n} s_i(A)}{\sum_{i=1}^{n} \frac{1}{i}} < \infty\right\}.$$

To conform with the definition of $\mathscr{C}_p$, $1 \leq p < \infty$, from now on we shall denote by $\mathscr{C}_\infty$ the ideal of all compact operators in $\mathscr{B}(H)$. If $A \in \mathscr{C}_1$ we denote the trace of A by $\operatorname{tr}(A)$.

Each of these ideals is determined by a symmetric norming (s.n.) function (for the definition of a s.n. function see [8], page 77). The corresponding s.n. functions to these ideals are

$$\phi_p(\xi) = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}, \quad 1 \leq p < \infty, \qquad \phi_\omega(\xi) = \sum_{i=1}^{\infty} \frac{1}{i} |\xi_{n_i}|$$

and

$$\phi_\Omega(\xi) = \sup_n \frac{\sum_{i=1}^{n} |\xi_{n_i}|}{\sum_{i=1}^{n} \frac{1}{i}}$$

where $\xi = \{\xi_i\}$ is any sequence of real numbers tending to zero and $\{n_i\}$ is a permutation of the positive integers such that $\{|\xi_{n_i}|\}$ is non-increasing. The ideals $\mathscr{C}_p$, $\mathscr{C}_\omega$, $\mathscr{C}_\Omega$, are Banach spaces under the corresponding norms

$$\|A\|_p = \left(\sum_{i=1}^{\infty} s_i(A)^p\right)^{1/p}, \qquad \|A\|_\omega = \sum_{i=1}^{\infty} \frac{1}{i} s_i(A), \qquad \|A\|_\Omega = \sup_n \frac{\sum_{i=1}^{n} s_i(A)}{\sum_{i=1}^{n} \frac{1}{i}}.$$

In the previous section we studied the convergence of the module triangular integral of an operator $A \in \mathscr{B}(H)$ in the uniform operator topology of $\mathscr{B}(H)$. If $\mathscr{P} = \{E_i : 1 \leq i \leq n\}$ is a partition of $\mathscr{E}$ and $E_{i-1} \leq F_i \leq E_i$, $i = 1, 2, \ldots, n$ then if $\tilde{\mathfrak{J}}(A)$ exists, as we know

$$\tilde{\mathfrak{J}}(A) = \lim_{\mathscr{P}} \sum_{i=1}^{n} \tilde{F}_i A \Delta E_i$$

and it is clear that $\tilde{\mathscr{L}}_{\mathscr{P}}(A)$ and $\tilde{\mathscr{U}}_{\mathscr{P}}(A)$ converge to a common limit; or equivalently $\tilde{\mathfrak{D}}_{\mathscr{P}}(A)$ converges to zero and one of $\tilde{\mathscr{L}}_{\mathscr{P}}(A)$, $\tilde{\mathscr{U}}_{\mathscr{P}}(A)$ converges. The converse of this is also true in the uniform operator topology and it follows from the inequality

$$\left\|\sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1}) A \Delta E_i\right\| \leq \|\tilde{\mathfrak{D}}_{\mathscr{P}}(A)\| \quad \text{or} \quad \left\|\sum_{i=1}^{n} (\tilde{E}_i - \tilde{F}_i) A \Delta E_i\right\| \leq \|\tilde{\mathfrak{D}}_{\mathscr{P}}(A)\|.$$

We shall show that this converse remains true on any symmetrically normed ideal. Indeed, let $\mathscr{C}_\phi$ be any symmetrically normed ideal determined by the s.n. function ϕ, and suppose that for $X \in \mathscr{C}_\phi$, $\tilde{\mathscr{L}}_{\mathscr{P}}(X)$ converges in $\mathscr{C}_\phi$ and $\tilde{\mathfrak{D}}_{\mathscr{P}}(X)$ tends to zero in the symmetric norm $\|\cdot\|_\phi$. We have

$$\sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1}) X \Delta E_i = \sum_{j=1}^{n} (\tilde{F}_j - \tilde{E}_{j-1}) \tilde{\mathfrak{D}}_{\mathscr{P}}(X).$$

If $\{s_k(T)\}$ denotes the set of characteristic numbers of an operator T, then by $s_k(AT) \leq \|A\| s_k(T)$, $k = 1, 2, \ldots$ for any $A \in \mathcal{B}(H)$, we get

$$s_k\left(\sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1}) X \Delta E_i\right) = s_k\left(\sum_{j=1}^{n} (\tilde{F}_j - \tilde{E}_{j-1}) \tilde{\mathfrak{D}}_{\mathcal{P}}(X)\right)$$

$$\leq \left\|\sum_{j=1}^{n} (\tilde{F}_j - \tilde{E}_{j-1})\right\| s_k(\tilde{\mathfrak{D}}_{\mathcal{P}}(X))$$

$$\leq s_k(\tilde{\mathfrak{D}}_{\mathcal{P}}(X)).$$

Put for convenience, $s'_k = s_k\left(\sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1}) X \Delta E_i\right)$ and $s''_k = s_k(\tilde{\mathfrak{D}}_{\mathcal{P}}(X))$. Then

$$\phi(s'_1, s'_2, \ldots, s'_r, 0, \ldots) \leq \phi(s''_1, s''_2, \ldots, s''_r, 0, \ldots)$$

for all r and so $\left\|\sum_{i=1}^{n} (\tilde{F}_i - \tilde{E}_{i-1}) X \Delta E_i\right\|_\phi \leq \|\tilde{\mathfrak{D}}_{\mathcal{P}}(X)\|_\phi$ which clearly implies the existence of the module triangular integral $\tilde{\mathfrak{T}}(X)$ in $\mathcal{C}_\phi$.

As we have mentioned we are going to study the module triangular integral on symmetrically normed ideals and the previous discussion enables us to confine our attention to the convergence or otherwise of $\tilde{\mathfrak{L}}_{\mathcal{P}}(A)$, $\tilde{\mathfrak{U}}_{\mathcal{P}}(A)$ and $\tilde{\mathfrak{D}}_{\mathcal{P}}(A)$.

As $\mathcal{P}$ varies, the set $\{\tilde{\mathfrak{L}}_{\mathcal{P}}\}$ forms an increasing net of projections on $\mathcal{B}(H)$ in the sense that if $\mathcal{P}_1$ is a refinement of $\mathcal{P}$ then $\tilde{\mathfrak{L}}_{\mathcal{P}_1} > \tilde{\mathfrak{L}}_{\mathcal{P}}$. That is, $\tilde{\mathfrak{L}}_{\mathcal{P}} = \tilde{\mathfrak{L}}_{\mathcal{P}} \tilde{\mathfrak{L}}_{\mathcal{P}_1} = \tilde{\mathfrak{L}}_{\mathcal{P}_1} \tilde{\mathfrak{L}}_{\mathcal{P}}$.

Similarly $\{\tilde{\mathfrak{U}}_{\mathcal{P}}\}$ and $\{\tilde{\mathfrak{D}}_{\mathcal{P}}\}$ are both decreasing nets of projections on $\mathcal{B}(H)$. In the sequel we shall prove that these nets automatically converge on the $\mathcal{C}_p$, $1 < p < \infty$, classes, and that for any operator A in the symmetrically normed ideal $\mathcal{C}_\omega$, the nets $\{\tilde{\mathfrak{L}}_{\mathcal{P}}(A)\}$, $\{\tilde{\mathfrak{U}}_{\mathcal{P}}(A)\}$ and $\{\tilde{\mathfrak{D}}_{\mathcal{P}}(A)\}$ converge in $\mathcal{B}(H)$. We shall make use of the following two theorems. The first one is concerned with the limits of an increasing or decreasing sequence of uniformly bounded projections on a reflexive Banach space (see, for example, [2]) and the second one describes the dual spaces of the symmetrically normed ideals that are used in this section.

Theorem 15. *If $\{P_\gamma : \gamma \in \Gamma\}$ is a uniformly bounded increasing (respectively decreasing) net of projections on a reflexive Banach space, then it converges in the strong operator topology to its supremum (respectively infimum).*

Theorem 16. *The dual spaces of $\mathcal{C}_p$ $(1 < p < \infty)$, $\mathcal{C}_1$ and $\mathcal{C}_\omega$ are respectively isometrically isomorphic to $\mathcal{C}_q$ $(1/p + 1/q = 1)$, $\mathcal{B}(H)$ and $\mathcal{C}_\Omega$. A conjugate linear isomorphism is given by $A \leftrightarrow f_A$ where*

$$f_A(X) = \operatorname{tr}(A^* X).$$

We are now ready to characterize the adjoints of $\tilde{\mathfrak{L}}_{\mathcal{P}}$, $\tilde{\mathfrak{U}}_{\mathcal{P}}$, and $\tilde{\mathfrak{D}}_{\mathcal{P}}$. The following lemma is the analogue of Lemma 2.3, [4].

Lemma 17. *If $\mathcal{C}$ is one of the ideals defined above the adjoints of $\tilde{\mathfrak{L}}_{\mathcal{P}}$, $\tilde{\mathfrak{U}}_{\mathcal{P}}$ and $\tilde{\mathfrak{D}}_{\mathcal{P}}$ regarded as operators on $\mathcal{C}$ are respectively $\tilde{\mathfrak{L}}_{\mathcal{P}}$, $\tilde{\mathfrak{U}}_{\mathcal{P}}$ and $\tilde{\mathfrak{D}}_{\mathcal{P}}$ regarded as operators on $\mathcal{C}^*$ where $\mathcal{C}^*$ is the dual of the ideal $\mathcal{C}$.*

Proof. To prove this we use the fact that if $XY \in \mathscr{C}_1$ then $\operatorname{tr}(XY) = \operatorname{tr}(YX)$. Let $\mathscr{P} = \{E_i : 1 \leq i \leq n\}$ be a partition of $\mathscr{E}$ and let $A \in \mathscr{C}$, $B \in \mathscr{C}^*$. Then,

$$\begin{aligned}\tilde{\mathscr{L}}_{\mathscr{P}}{}^*(f_B)(A) &= f_B(\tilde{\mathscr{L}}_{\mathscr{P}}(A)) = \operatorname{tr}\left(B^* \sum_{i=1}^{n} \tilde{E}_{i-1} A \Delta E_i\right)\\ &= \sum_{n=1}^{n} \operatorname{tr}(B^* \tilde{E}_{i-1} A \Delta E_i)\\ &= \sum_{i=1}^{n} [\operatorname{tr}((\tilde{E}_{i-1} B \Delta E_i)^* A)]\\ &= \operatorname{tr}\left[\left(\sum_{i=1}^{n} \tilde{E}_{i-1} B \Delta E_i\right)^* A\right]\\ &= \operatorname{tr}(\tilde{\mathscr{L}}_{\mathscr{P}}(B)^* A) = f_{\tilde{\mathscr{L}}(B)}(A).\end{aligned}$$

Hence $\tilde{\mathscr{L}}_{\mathscr{P}}{}^*(f_B) = f_{\tilde{\mathscr{L}}_{\mathscr{P}}(B)}$ which means that the adjoint of the operator $\tilde{\mathscr{L}}_{\mathscr{P}}$ on $\mathscr{C}$ is $\tilde{\mathscr{L}}_{\mathscr{P}}$ regarded as an operator on $\mathscr{C}^*$. Similarly we identify the adjoints of the operators $\tilde{\mathscr{U}}_{\mathscr{P}}$ and $\tilde{\mathscr{D}}_{\mathscr{P}}$.

Corollary 18. *The operators $\tilde{\mathscr{L}}_{\mathscr{P}}$, $\tilde{\mathscr{U}}_{\mathscr{P}}$ and $\tilde{\mathscr{D}}_{\mathscr{P}}$ are self-adjoint projections on the Hilbert space $\mathscr{C}_2$ and therefore*

$$\|\tilde{\mathscr{L}}_{\mathscr{P}}\|_2 = \|\tilde{\mathscr{U}}_{\mathscr{P}}\|_2 = \|\tilde{\mathscr{D}}_{\mathscr{P}}\|_2 = 1.$$

Proof. Immediate from Lemma 17.

The following set of identities is the main tool for the proof of the convergence of the nets $\{\tilde{\mathscr{L}}_{\mathscr{P}}]$, $\{\tilde{\mathscr{U}}_{\mathscr{P}}]$ and $\{\tilde{\mathscr{D}}_{\mathscr{P}}]$ in the norm of $\mathscr{C}_p$ $(1 < p < \infty)$.

Lemma 19. *Let $A \in \mathscr{B}(H)$. For any nest $\mathscr{E}$ and any partition $\mathscr{P}$ of $\mathscr{E}$ the following identities are valid:*

(*i*) $\tilde{\mathscr{D}}_{\mathscr{P}}(A)^* \tilde{\mathscr{D}}_{\mathscr{P}}(A) = \tilde{\mathscr{D}}_{\mathscr{P}}(A^* \tilde{\mathscr{D}}_{\mathscr{P}}(A))$

(*ii*) $\tilde{\mathscr{L}}_{\mathscr{P}}(A)^* \tilde{\mathscr{L}}_{\mathscr{P}}(A) = \tilde{\mathscr{L}}_{\mathscr{P}}(\tilde{\mathscr{L}}_{\mathscr{P}}(A)^* A) + (\tilde{\mathscr{U}}_{\mathscr{P}}(\tilde{\mathscr{L}}_{\mathscr{P}}(A)^* A))^*$

(*iii*) $\tilde{\mathscr{U}}_{\mathscr{P}}(A)^* \tilde{\mathscr{U}}_{\mathscr{P}}(A) = \tilde{\mathscr{L}}_{\mathscr{P}}(\tilde{\mathscr{U}}_{\mathscr{P}}(A)^* A) + (\tilde{\mathscr{U}}_{\mathscr{P}}(\tilde{\mathscr{U}}_{\mathscr{P}}(A)^* A))^*$.

Proof. Since the proof of (i) is easy and the proofs of (ii) and (iii) are based on similar manipulations, only the proof of (ii) is given. For this we use the fact that $E_i \leq E_j$ implies $\tilde{E}_i \leq \tilde{E}_j$ and

$$E_i \Delta E_j = \begin{cases} \Delta E_j & \text{for } i \geq j \\ 0 & \text{for } i < j. \end{cases}$$

The result now comes from a series of simple calculations. Indeed,

$$\begin{aligned}\tilde{\mathscr{L}}_{\mathscr{P}}(A)^* \tilde{\mathscr{L}}_{\mathscr{P}}(A) &= \left(\sum_{i=1}^{n} \Delta E_i A^* \tilde{E}_{i-1}\right)\left(\sum_{j=1}^{n} \tilde{E}_{j-1} A \Delta E_j\right)\\ &= \sum_{i=1}^{n} E_{i-1}\left(\sum_{j=1}^{n} \Delta E_j A^* \tilde{E}_{j-1}\right) A \Delta E_i + \sum_{i=1}^{n} \Delta E_i A^* \left(\sum_{j=1}^{n} \tilde{E}_{j-1} A \Delta E_j\right) E_i\end{aligned}$$

$$= \sum_{i=1}^{n} E_{i-1}\left(\sum_{j=1}^{n} \tilde{E}_{j-1} A \Delta E_j\right)^* A \Delta E_i + \left[\sum_{i=1}^{n} E_i\left(\sum_{j=1}^{n} \tilde{E}_{j-1} A \Delta E_j\right)^* A \Delta E_i\right]^*$$

$$= \mathcal{L}_{\mathcal{P}}[\tilde{\mathcal{L}}_{\mathcal{P}}(A)^*A] + (\mathcal{U}_{\mathcal{P}}[\tilde{\mathcal{L}}_{\mathcal{P}}(A)^*A])^*.$$

Now to prove one of the main results of this section, namely the convergence of the nets $\{\tilde{\mathcal{L}}_{\mathcal{P}}\}$, $\{\tilde{\mathcal{U}}_{\mathcal{P}}\}$ and $\{\tilde{\mathcal{D}}_{\mathcal{P}}\}$ on the $\mathcal{C}_p$ $(1 < p < \infty)$ classes we shall use the following theorem which is essentially Theorem 3.2 in [4].

Theorem 20. *If $1 < p < \infty$, then for any nest, the nets $\{\mathcal{L}_{\mathcal{P}}\}$, $\{\mathcal{U}_{\mathcal{P}}\}$ and $\{\mathcal{D}_{\mathcal{P}}\}$ are uniformly bounded, considering them as operators on $\mathcal{C}_p$.*

Theorem 21. *If $1 < p < \infty$, then for any nest $\mathcal{E}$, the nets $\{\tilde{\mathcal{L}}_{\mathcal{P}}\}$, $\{\tilde{\mathcal{U}}_{\mathcal{P}}\}$ and $\{\tilde{\mathcal{D}}_{\mathcal{P}}\}$ converge strongly to bounded operators $\tilde{\mathcal{L}}$, $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{D}}$ on the space $\mathcal{C}_p$.*

Proof. We know from Theorem 16 that $\mathcal{C}_p$ $(1 < p < \infty)$ is reflexive. Therefore from Theorem 15 it will be sufficient to prove that $\{\tilde{\mathcal{L}}_{\mathcal{P}}\}$, $\{\tilde{\mathcal{U}}_{\mathcal{P}}\}$ and $\{\tilde{\mathcal{D}}_{\mathcal{P}}\}$ are uniformly bounded. We shall use the fact that if X, $Y \in \mathcal{C}_p$ then $\|X\|_p = \|X^*\|_p$, $\|XY\|_p \leq \|X\|_{2p}\|Y\|_{2p}$ and $\|X\|_{2p}^2 = \|X^*X\|_p$. Also if $\mathcal{T}$ is an operator on $\mathcal{C}_p$ the operator norm of $\mathcal{T}$ will be denoted by $\|\mathcal{T}\|_p$.

From Lemma 19(i), for all $A \in \mathcal{C}_{2p}$ we have

$$\begin{aligned}
\|\tilde{\mathcal{D}}_{\mathcal{P}}(A)\|_{2p}^2 &= \|\tilde{\mathcal{D}}_{\mathcal{P}}(A)^*\tilde{\mathcal{D}}_{\mathcal{P}}(A)\|_p \\
&= \|\mathcal{D}_{\mathcal{P}}(A^*\tilde{\mathcal{D}}_{\mathcal{P}}(A))\|_p \\
&\leq \|\mathcal{D}_{\mathcal{P}}\|_p\|A^*\tilde{\mathcal{D}}_{\mathcal{P}}(A)\|_p \\
&\leq \|\mathcal{D}_{\mathcal{P}}\|_p\|A^*\|_{2p}\|\tilde{\mathcal{D}}_{\mathcal{P}}(A)\|_{2p} \\
&= \|\mathcal{D}_{\mathcal{P}}\|_p\|A\|_{2p}\|\tilde{\mathcal{D}}_{\mathcal{P}}(A)\|_{2p}.
\end{aligned}$$

Dividing by $\|A\|_{2p}\|\tilde{\mathcal{D}}_{\mathcal{P}}(A)\|_{2p}$ and taking the supremum over all $A \in \mathcal{C}_{2p}$ we get

$$\|\tilde{\mathcal{D}}_{\mathcal{P}}\|_{2p} \leq \|\mathcal{D}_{\mathcal{P}}\|_p. \tag{7}$$

Similarly from Lemma 19(ii), for all $A \in \mathcal{C}_{2p}$,

$$\begin{aligned}
\|\tilde{\mathcal{L}}_{\mathcal{P}}(A)\|_{2p}^2 &= \|\tilde{\mathcal{L}}_{\mathcal{P}}(A)^*\tilde{\mathcal{L}}_{\mathcal{P}}(A)\|_p \\
&= \|\mathcal{L}_{\mathcal{P}}[\tilde{\mathcal{L}}_{\mathcal{P}}(A)^*A] + (\mathcal{U}_{\mathcal{P}}[\tilde{\mathcal{L}}_{\mathcal{P}}(A)^*A])^*\|_p \\
&\leq \|\mathcal{L}_{\mathcal{P}}\|_p\|\tilde{\mathcal{L}}_{\mathcal{P}}(A)^*A\|_p + \|\mathcal{U}_{\mathcal{P}}\|_p\|\tilde{\mathcal{L}}_{\mathcal{P}}(A)^*A\|_p \\
&\leq (\|\mathcal{L}_{\mathcal{P}}\|_p + \|\mathcal{U}_{\mathcal{P}}\|_p)\|\tilde{\mathcal{L}}_{\mathcal{P}}(A)\|_{2p}\|A\|_{2p}.
\end{aligned}$$

Hence, as above, dividing by $\|\tilde{\mathcal{L}}_{\mathcal{P}}(A)\|_{2p}\|A\|_{2p}$ and taking the supremum, we get

$$\|\tilde{\mathcal{L}}_{\mathcal{P}}\|_{2p} \leq \|\mathcal{L}_{\mathcal{P}}\|_p + \|\mathcal{U}_{\mathcal{P}}\|_p. \tag{8}$$

Finally from Lemma 19(iii) with the same argument we have,

$$\|\tilde{\mathcal{U}}_{\mathcal{P}}\|_{2p} \leq \|\mathcal{L}_{\mathcal{P}}\|_p + \|\mathcal{U}_{\mathcal{P}}\|_p. \tag{9}$$

Now from Theorem 20, Corollary 18 and the inequalities (7), (8) and (9) we have that for $2 \le p < \infty$, the nets $\{\tilde{\mathcal{L}}_{\mathcal{P}}\}$, $\{\tilde{\mathcal{U}}_{\mathcal{P}}\}$ and $\{\tilde{\mathcal{D}}_{\mathcal{P}}\}$ are uniformly bounded. The same result follows for $1 < p \le 2$ by duality using Lemma 17. (Note that $\mathcal{C}_p^* = \mathcal{C}_q$, $q = p/(p-1)$ and $q \ge 2$ implies $1 < p \le 2$.) The proof now is completed by Theorem 15.

Next we shall prove the existence of the integral $\tilde{\mathcal{D}}(K)$ for any compact operator $K \in \mathcal{C}_\infty$ in a different way from that in Proposition 12. This proof depends on Theorem 21.

Theorem 22. *For any nest $\mathcal{E}$ and any partition $\mathcal{P}$ of $\mathcal{E}$, $\|\tilde{\mathcal{D}}_{\mathcal{P}}\|_\infty \le 1$ and the net $\{\tilde{\mathcal{D}}_{\mathcal{P}}\}$ converges on $\mathcal{C}_\infty$.*

Proof. Let $\mathcal{P} = \{E_i : 1 \le i \le n\}$ be a partition for $\mathcal{E}$ and x be any non-zero vector in H. Then $\sum_{i=1}^n \|\Delta E_i x\|^2 = \|x\|^2$ and for any $A \in \mathcal{B}(H)$,

$$\begin{aligned}
\|\tilde{\mathcal{D}}_{\mathcal{P}}(A)x\|^2 &= \sum_{i=1}^n \|\Delta \tilde{E}_i A \Delta E_i x\|^2 \\
&\le (\max_{1\le i\le n} \|\Delta \tilde{E}_i A \Delta E_i\|^2) \sum_{i=1}^n \|\Delta E_i x\|^2 \\
&\le \|A\|^2 \|x\|^2.
\end{aligned}$$

Hence $\|\tilde{\mathcal{D}}_{\mathcal{P}}\|_\infty \le 1$. Let $A \in \mathcal{C}_\infty$ and $X \in \mathcal{C}_2$. Then for any two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\mathcal{E}$, we have

$$\begin{aligned}
\|\tilde{\mathcal{D}}_{\mathcal{P}_1}(A) - \tilde{\mathcal{D}}_{\mathcal{P}_2}(A)\| &\le \|\tilde{\mathcal{D}}_{\mathcal{P}_1}(A - X)\| + \|\tilde{\mathcal{D}}_{\mathcal{P}_1}(X) - \tilde{\mathcal{D}}_{\mathcal{P}_2}(X)\| + \|\tilde{\mathcal{D}}_{\mathcal{P}_2}(X - A)\| \\
&\le 2\|A - X\| + \|\tilde{\mathcal{D}}_{\mathcal{P}_1}(X) - \tilde{\mathcal{D}}_{\mathcal{P}_2}(X)\|_2.
\end{aligned}$$

Since, by Theorem 21, $\{\tilde{\mathcal{D}}_{\mathcal{P}}\}$ converges strongly in $\mathcal{C}_2$ and $\mathcal{C}_2$ is dense in $\mathcal{C}_\infty$ it follows that $\{\tilde{\mathcal{D}}_{\mathcal{P}}(A)\}$ is a Cauchy net in $\mathcal{C}_\infty$ and hence converges.

Finally in this section we shall prove that the nets $\{\tilde{\mathcal{L}}_{\mathcal{P}}(A)\}$, $\{\tilde{\mathcal{U}}_{\mathcal{P}}(A)\}$ and $\{\tilde{\mathcal{D}}_{\mathcal{P}}(A)\}$ converge in $\mathcal{B}(H)$ for any $A \in \mathcal{C}_\omega$. The class $\mathcal{C}_\omega$ which was introduced by I. V. Macaev with norm slightly different from that in [4], (Macaev used the sequence $1/(2i-1)$ instead of $1/i$), plays an important role in the theory of triangular integrals. For example the condition $X \in \mathcal{C}_\omega$ is necessary and sufficient for the convergence of the triangular integral $\mathcal{T}(X) = \int_{\mathcal{E}} EXdE$ for every continuous nest $\mathcal{E}$. (See Theorem 4.1, Ch. III, [8].)

J. A. Erdos in [4] gave new simpler proofs for the convergence of the nets $\{\tilde{\mathcal{L}}_{\mathcal{P}}(A)\}$, $\{\tilde{\mathcal{U}}_{\mathcal{P}}(A)\}$ and $\{\tilde{\mathcal{D}}_{\mathcal{P}}(A)\}$, $A \in \mathcal{C}_\omega$ and we shall use his techniques to prove our results. It should be mentioned here that in the proof of Lemma 3.2 Section 4, [4] there are some misprints. Namely instead of $G_k = I$ and $S = \sum_{i=1}^k G_i R(F_i - F_{i-1})$ we should have $G_{k+1} = I$ and $S = \sum_{i=1}^k F_i R(G_{i+1} - G_i)$ and subsequently all the summations and maxima should be taken from 1 to $k + 1$.

The following lemma establishes bounds for the characteristic numbers of the image of a rank one operator under $\tilde{\mathcal{U}}_{\mathcal{P}}$ and it is the main tool for proving the convergence theorem.

Lemma 23. *If $\mathcal{P}$ is a partition of any nest $\tilde{\mathcal{E}}$ and R is a rank one operator of norm one then the characteristic numbers of $\tilde{\mathcal{U}}_{\mathcal{P}}(R)$ satisfy the conditions*

$$s_{r+1}[\tilde{\mathcal{U}}_{\mathcal{P}}(R)] \leq \frac{1}{r}, \qquad r = 1, 2, \ldots .$$

Proof. Suppose $\mathcal{P} = \{E_i : 0 \leq i \leq n\}$ and let $\tilde{\mathcal{P}} = \{Q_1, Q_2, \ldots, Q_m\}$, $m \leq n$ be the set of distinct images of the members of $\mathcal{P}$ under the order homomorphism $E \rightarrow \tilde{E}$. Put

$$\mathcal{P}_1 = \{E_j \in \mathcal{P} : \tilde{E}_j = Q_j, 1 \leq j \leq m \text{ and for any } E_i \in \mathcal{P} : \tilde{E}_i = Q_j \Rightarrow E_i \leq E_j\}.$$

In other words for each $j : 1 \leq j \leq m$, E_j is the largest member of $\mathcal{P}$ with the property $\tilde{E}_j = Q_j$. We choose a subset $\{F_j\}$ of $\mathcal{P}_1$ inductively as follows.

Let F_0 be the projection in $\mathcal{P}_1$ with $\tilde{F}_0 = 0$ and let F_j be the smallest member P of $\mathcal{P}_1$ such that

$$\|(\tilde{P} - \tilde{F}_{j-1})R(P - F_{j-1})\| \geq \frac{1}{r}.$$

If F_k is the largest member of $\{F_j\}$ then

$$\|(\tilde{I} - \tilde{F}_k)R(I - F_k)\| < \frac{1}{r}.$$

For each j the operator $(\tilde{F}_j - \tilde{F}_{j-1})R(F_j - F_{j-1})$ has rank one and so there exist unit vectors x_j, y_j with

$$(F_j - F_{j-1})x_j = x_j, \qquad (\tilde{F}_j - \tilde{F}_{j-1})y_j = y_j$$

and

$$|\langle(\tilde{F}_j - \tilde{F}_{j-1})R(F_j - F_{j-1})x_j, y_j\rangle| = \|(\tilde{F}_j - \tilde{F}_{j-1})R(F_j - F_{j-1})\| \geq \frac{1}{r}.$$

Hence as R has rank one

$$1 = \|R\| = \|R\|_1 \geq \sum_{j=1}^{k} |\langle(\tilde{F}_j - \tilde{F}_{j-1})R(F_j - F_{j-1})x_j, y_j\rangle| \geq \frac{k}{r}$$

and so $k \leq r$. Let G_j be the member of $\mathcal{P}_1$ immediately preceding F_j and put $G_{k+1} = I$. Then from the definition of F_j it is clear that

$$\|(\tilde{G}_j - \tilde{F}_{j-1})R(G_j - F_{j-1})\| \leq \frac{1}{r}, \qquad j = 1, 2, \ldots, k+1. \tag{10}$$

Define

$$S = \sum_{j=1}^{k} \tilde{F}_j R(G_{j+1} - G_j).$$

Then S has rank at most r and using a matrix calculation we can work out that

$$\tilde{\mathcal{U}}_{\mathcal{P}}(R) - S = \tilde{\mathcal{U}}_{\mathcal{P}}\left[\sum_{j=1}^{k+1} (\tilde{G}_j - \tilde{F}_{j-1})R(G_j - F_{j-1})\right]$$
$$= \sum_{j=1}^{k+1} (\tilde{G}_j - \tilde{F}_{j-1})\tilde{\mathcal{U}}_{\mathcal{P}}(R)(G_j - F_{j-1}).$$

But $s_{r+1}(X) = \inf\{\|X - T\| : \mathrm{rank}(T) \le r\}$ (see [8], page 48). Hence

$$s_{r+1}[\tilde{\mathcal{U}}_{\mathcal{P}}(R)] \le \|\tilde{\mathcal{U}}_{\mathcal{P}}(R) - S\|$$

(11)
$$= \left\|\sum_{j=1}^{k+1} (\tilde{G}_j - \tilde{F}_{j-1})\tilde{\mathcal{U}}_{\mathcal{P}}(R)(G_j - F_{j-1})\right\|$$

$$\le \max_{1\le j\le k+1} \{\|(\tilde{G}_j - \tilde{F}_{j-1})\tilde{\mathcal{U}}_{\mathcal{P}}(R)(G_j - F_{j-1})\|\}.$$

From Corollary 18 we have $\|\tilde{\mathcal{U}}_{\mathcal{P}}\|_2 = 1$ and it is easy to see (by a matrix calculation, as above) that

$$(\tilde{G}_j - \tilde{F}_{j-1})\tilde{\mathcal{U}}_{\mathcal{P}}(R)(G_j - F_{j-1}) = \tilde{\mathcal{U}}_{\mathcal{P}}[(\tilde{G}_j - \tilde{F}_{j-1})R(G_j - F_{j-1})].$$

Using the fact that for any $T \in \mathcal{B}(H)$, $\|T\| \le \|T\|_2$ with equality when T is rank one, we have from (11)

$$s_{r+1}[\tilde{\mathcal{U}}_{\mathcal{P}}(R)] \le \max_{1\le j\le k+1} \{\|\tilde{\mathcal{U}}_{\mathcal{P}}[\tilde{G}_j - \tilde{F}_{j-1})R(G_j - F_{j-1})]\|\}$$
$$\le \max_{1\le j\le k+1} \|(\tilde{G}_j - \tilde{F}_{j-1})R(G_j - F_{j-1})\|_2$$
$$= \max_{1\le j\le k+1} \|(\tilde{G}_j - \tilde{F}_{j-1})R(G_j - F_{j-1})\|, \qquad \text{(using (10))}$$
$$\le \frac{1}{r}.$$

Theorem 24. *For any nest $\mathcal{E}$, the nets $\{\tilde{\mathcal{L}}_{\mathcal{P}}\}$. $\{\tilde{\mathcal{U}}_{\mathcal{P}}\}$ and $\{\tilde{\mathfrak{D}}_{\mathcal{P}}\}$ converge strongly to bounded operators from $\mathcal{C}_\omega$ to $\mathcal{B}(H)$.*

Proof. We know that $\|A\| \le \|A\|_\omega$ for every $A \in \mathcal{C}_\omega$. Hence from Theorem 22

$$\|\tilde{\mathfrak{D}}_{\mathcal{P}}(A)\| \le \|A\| \le \|A\|_\omega$$

and so $\|\tilde{\mathfrak{D}}_{\mathcal{P}}\| \le 1$ on $\mathcal{C}_\omega$. Moreover, as in Theorem 22, we can prove that $\{\tilde{\mathfrak{D}}_{\mathcal{P}}\}$ converges strongly on $\mathcal{C}_\omega$. Since $\tilde{\mathcal{L}}_{\mathcal{P}} = \tilde{\mathcal{U}}_{\mathcal{P}} - \tilde{\mathfrak{D}}_{\mathcal{P}}$ it will be sufficient for the proof of the theorem to prove the convergence of $\{\tilde{\mathcal{U}}_{\mathcal{P}}\}$.

Let $A \in \mathcal{C}_\omega$. Then $\tilde{\mathcal{U}}_{\mathcal{P}}(A)$ is compact and thus

$$\tilde{\mathcal{U}}_{\mathcal{P}}(A) = \sum_{i=1}^{\infty} s_i(\tilde{\mathcal{U}}_{\mathcal{P}}(A))x_i \otimes y_i$$

where $\{x_i\}$ and $\{y_i\}$ are orthogonal systems and $\|\tilde{\mathcal{U}}_{\mathcal{P}}(A)\| = s_1(\tilde{\mathcal{U}}_{\mathcal{P}}(A))$. Therefore

$$\begin{aligned}
\|\tilde{\mathcal{U}}_{\mathcal{P}}(A)\| &= \langle \tilde{\mathcal{U}}_{\mathcal{P}}(A)x_1, y_1\rangle \\
&= \operatorname{tr}((y_1 \otimes x_1)\tilde{\mathcal{U}}_{\mathcal{P}}(A)) \\
&= \sum_{i=1}^{n} [\operatorname{tr}((y_1 \otimes x_1)\tilde{E}_i A \Delta E_i)] \\
&= \sum_{i=1}^{n} [\operatorname{tr}(\Delta E_i (y_1 \otimes x_1)\tilde{E}_i A)] \\
&= \operatorname{tr}\left[\sum_{i=1}^{n} \Delta E_i (y_1 \otimes x_1)\tilde{E}_i A \right] \\
&= \operatorname{tr}(\tilde{\mathcal{U}}_{\mathcal{P}}(y_1 \otimes x_1)^* A) \qquad \text{(and using Theorem 16)} \\
&= f_{\tilde{\mathcal{U}}_{\mathcal{P}}(x_1 \otimes y_1)}(A) \\
&\leq \|\tilde{\mathcal{U}}_{\mathcal{P}}(x_1 \otimes y_1)\|_{\Omega} \|A\|_{\omega}.
\end{aligned}$$

But

$$\begin{aligned}
\|\tilde{\mathcal{U}}_{\mathcal{P}}(x_1 \otimes y_1)z\| &= \left\| \sum_{i=1}^{n} \tilde{E}_i (x_1 \otimes y_1)\Delta E_i z \right\| \\
&\leq \max_{1 \leq i \leq n} \{\|\Delta E_i x_1 \otimes \tilde{E}_i y_1\|\} \cdot \|z\|^2
\end{aligned}$$

for any $z \in H$ and so $\|\tilde{\mathcal{U}}_{\mathcal{P}}(x_1 \otimes y_1)\| = s_1(\tilde{\mathcal{U}}_{\mathcal{P}}(x_1 \otimes y_1)) \leq 1$. Hence, using Lemma 23,

$$\|\tilde{\mathcal{U}}_{\mathcal{P}}(x_1 \otimes y_1)\|_{\Omega} = \sup_n \frac{\sum_{i=1}^{n} s_i(\tilde{\mathcal{U}}_{\mathcal{P}}(x_1 \otimes y_1))}{\sum_{i=1}^{n} \frac{1}{i}} \leq 1$$

and thus $\|\tilde{\mathcal{U}}_{\mathcal{P}}(A)\| \leq \|A\|_{\omega}$.

Now repeating the same argument as in Theorem 22 and using the fact that $\mathcal{C}_2$ is dense in $\mathcal{C}_\omega$ we have that $\{\tilde{\mathcal{U}}_{\mathcal{P}}(A)\}$ is a Cauchy net in $\mathcal{C}_\omega$ and hence converges. This completes the proof.

Theorem 25. *(i) Let X by a compact operator. A necessary and sufficient condition for the module triangular integral $\tilde{\mathfrak{T}}(X)$ of X to exist for every continuous next $\mathcal{E}$ and for every weakly closed module associated with $\mathcal{E}$ is that $X \in \mathcal{C}_\omega$.*

(ii) If $X \in \mathcal{C}_\omega$ and $\mathcal{A}(\mathcal{E},\sim)$ is a module with $\mathcal{E}$ a complete nest such that

$\sum(\tilde{E} - \tilde{E}^-)X(E - E^-) = 0$ *for all* $E \in \mathscr{E}$, $E \neq E^-$ *then* $\tilde{\mathscr{T}}(X)$ *also exists.*

(iii) Let X *be an operator in* $\mathscr{C}_\omega$ *and* $\mathscr{A}(\mathscr{E},\sim)$ *any weakly closed module with* $\mathscr{E}$ *continuous. Then* $X = X_1 + X_2$ *where* $X_1 \in \mathscr{A}(\mathscr{E},\sim)$ *and* $X_2 \in \mathscr{A}(\mathscr{E}^\perp,\sim)$. *The same is true for any operator* $X \in \mathscr{C}_\omega$ *satisfying condition (ii).*

Proof. For any operator $X \in \mathscr{C}_\omega$, when $\mathscr{E}$ is continuous or when X satisfies condition (ii) we have $\tilde{\mathscr{D}}(X) = 0$. Therefore from Theorem 24, in both cases, $\tilde{\mathscr{T}}(X)$ exists and $X = \tilde{\mathscr{L}}(X) + (X - \tilde{\mathscr{U}}(X))$ where $\tilde{\mathscr{L}}(X) \in \mathscr{A}(\mathscr{E},\sim)$ and $X - \tilde{\mathscr{U}}(X) \in \mathscr{A}(\mathscr{E}^\perp,\sim)$. The necessity in (i) is immediate from the fact that the triangular integral $\mathscr{T}(X) = \int_\mathscr{E} EXdE$ exists for every continuous nest if and only if $X \in \mathscr{C}_\omega$.

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