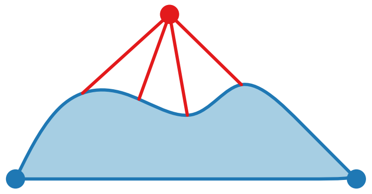
Visualisation of graphs

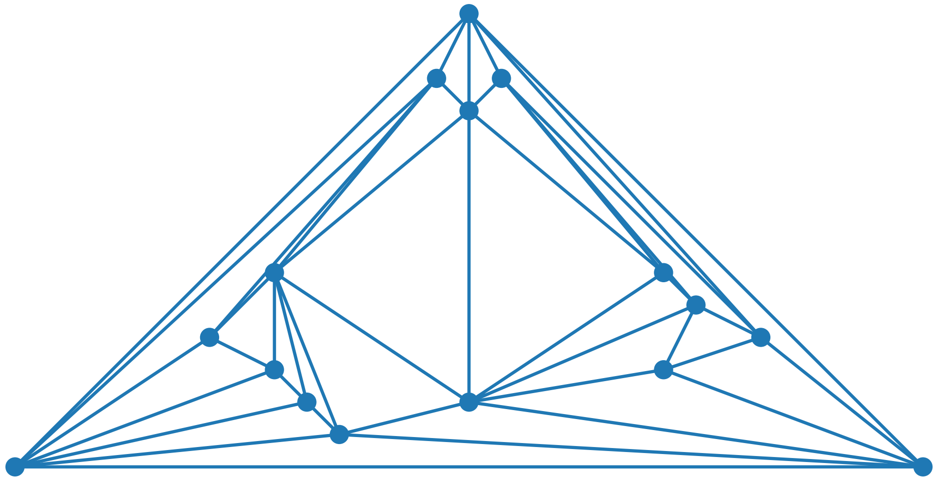
Planar straight-line drawings

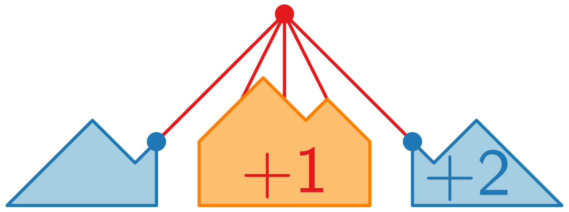
Canonical order

Antonios Symvonis · Chrysanthi Raftopoulou

Fall semester 2020







The original slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, ...
The original presentation was modified/updated by A. Symvonis and C. Raftopoulou

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3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

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- crossings reduce readability
- bends reduce readability

Planar graphs

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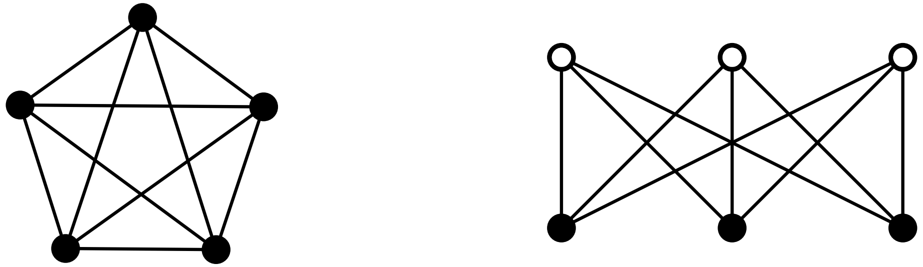

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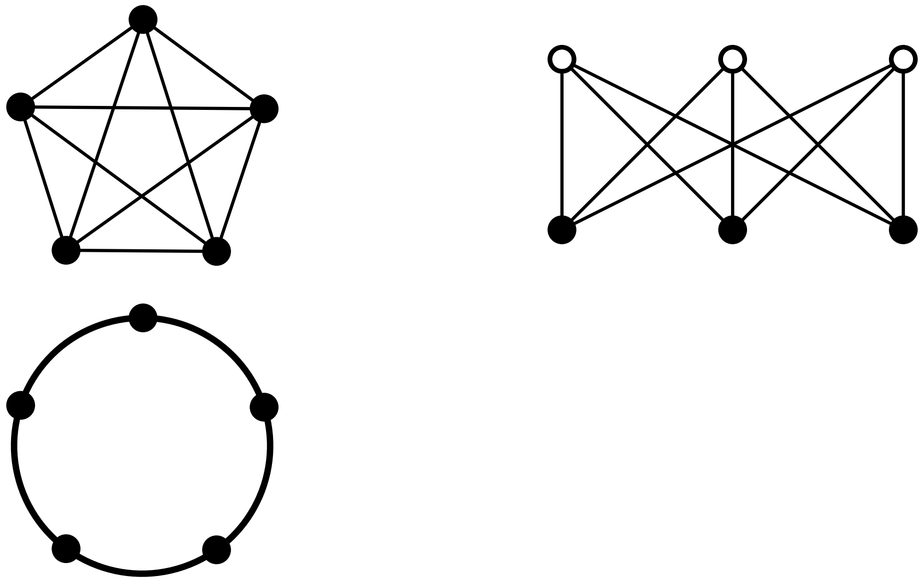

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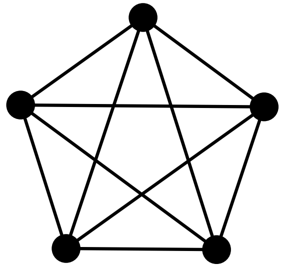

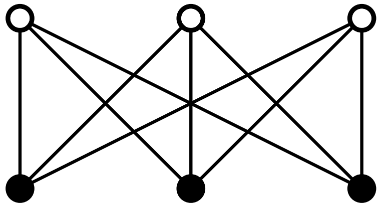

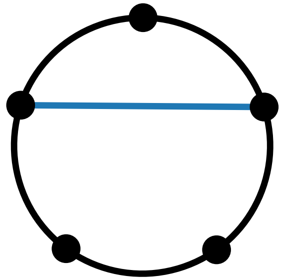

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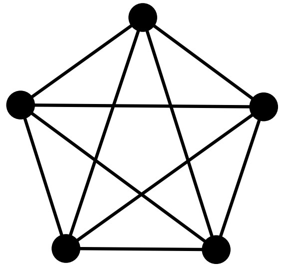

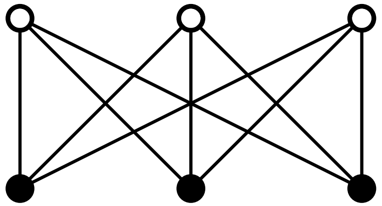

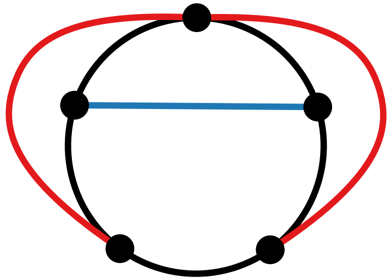

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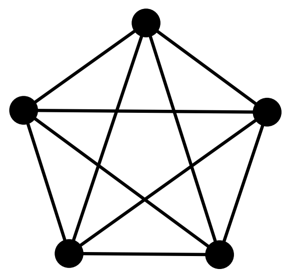

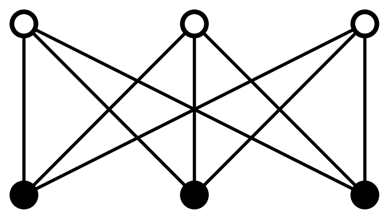

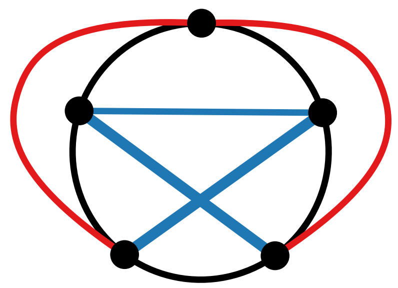

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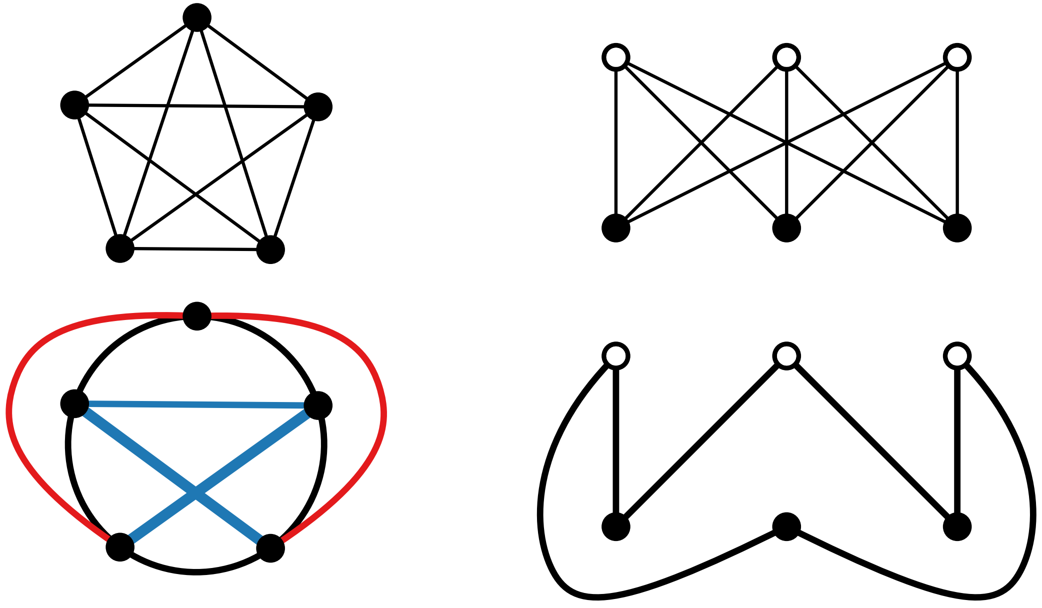

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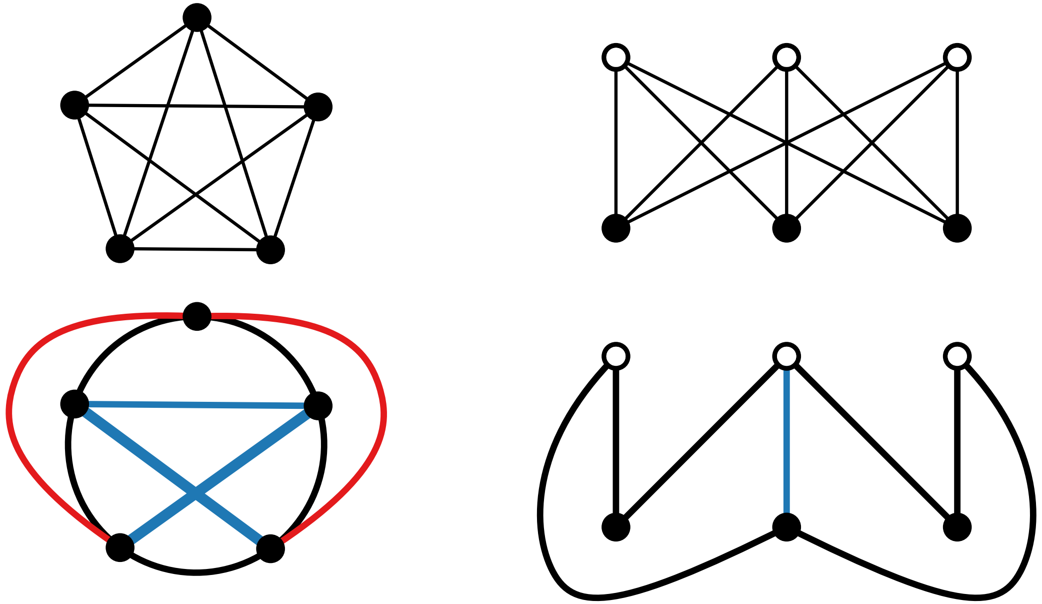

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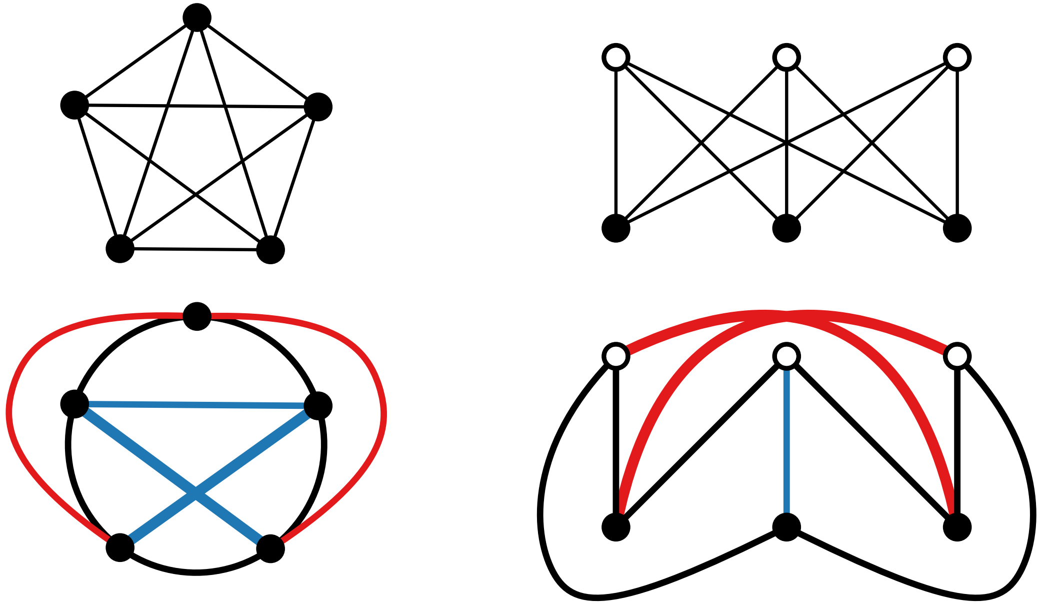

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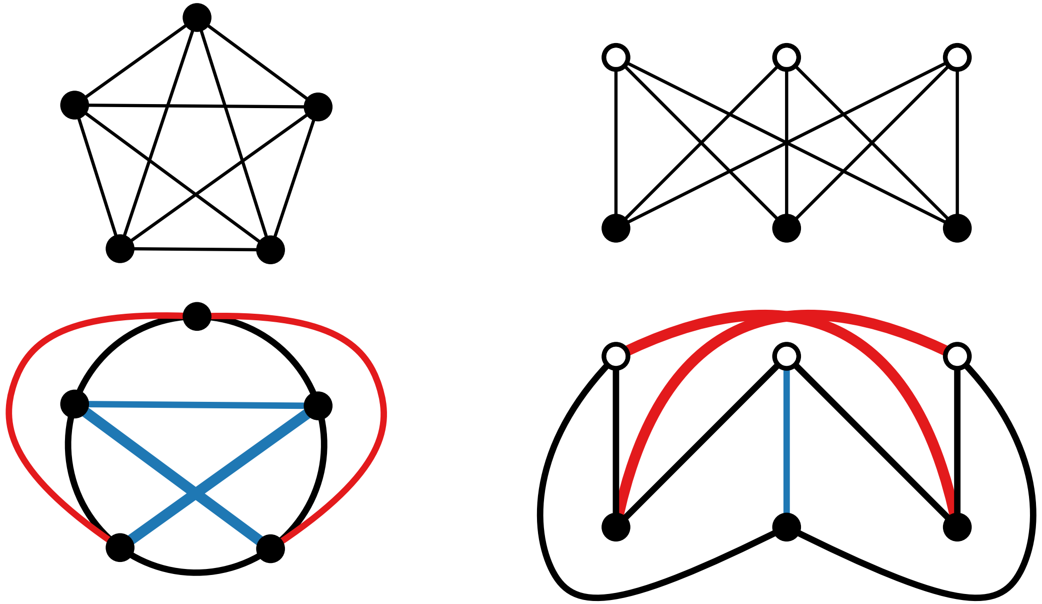

- **Recognition:** For a graph G with n vertices, there is an $\mathcal{O}(n)$ time algorithm to test if G is planar. [Hopcroft & Tarjan 1974]
  - Also computes an *embedding* in $\mathcal{O}(n)$.

Planar graphs

- **Embedding of planar graph:**
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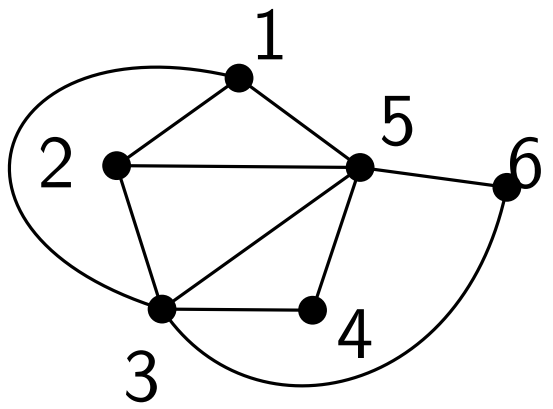

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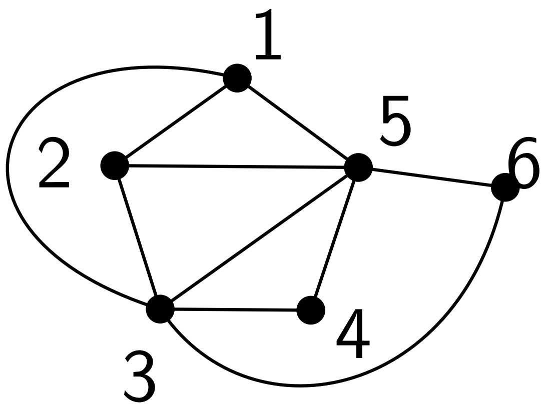
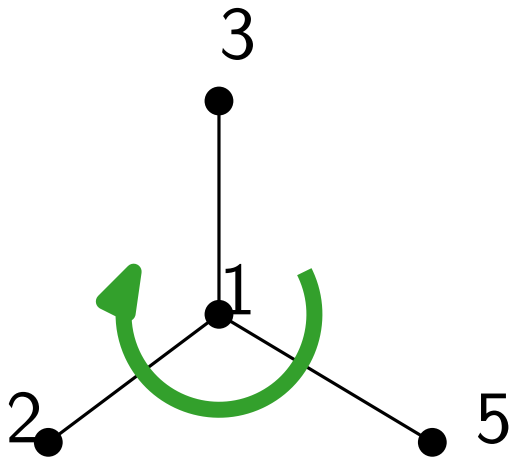


- **Edges:**
  $1 : \{(1,5),(1,2),(1,3)\}$



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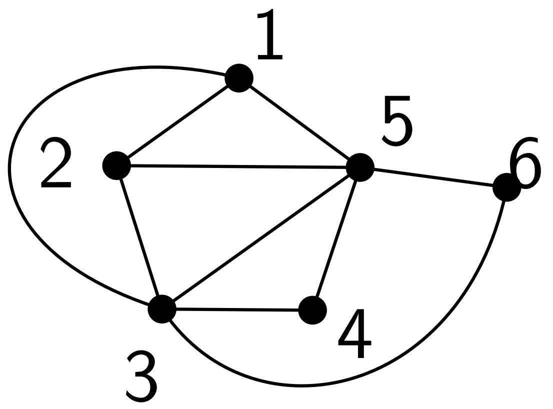

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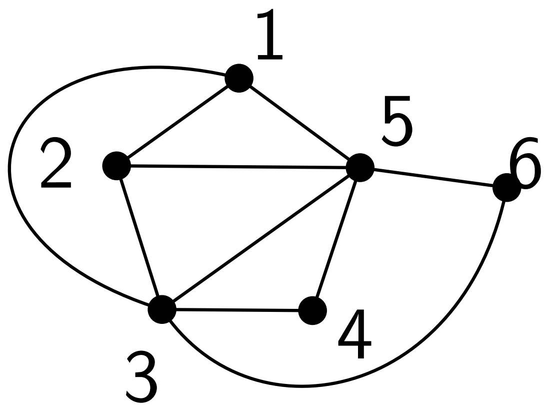

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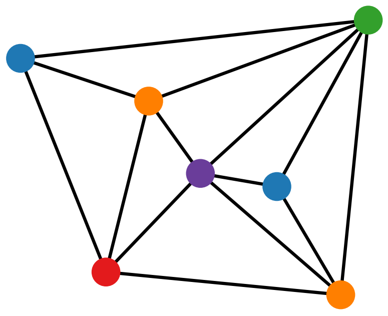

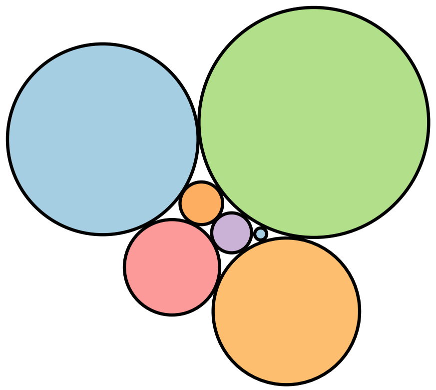

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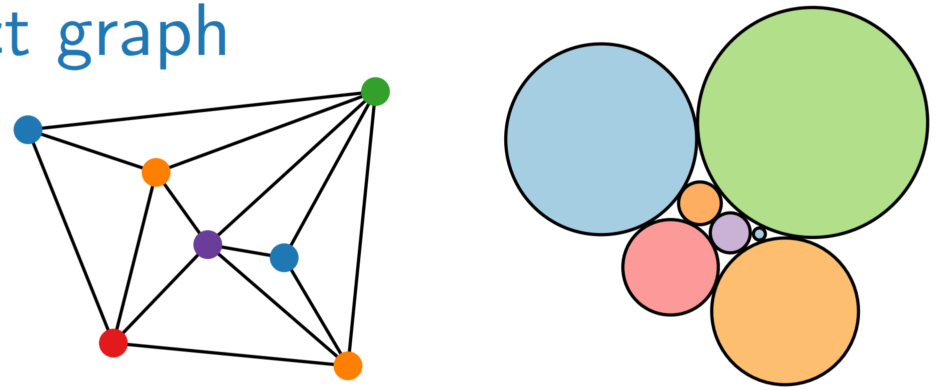

- Every 3-connected planar graph has an embedding with convex polygons as its faces (i.e., implies straight lines). [Tutte 1963: How to draw a graph]
  - Idea: Place vertices in the barycentre of neighbours.
  - Drawback: Requires large grids.

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  Exponential area

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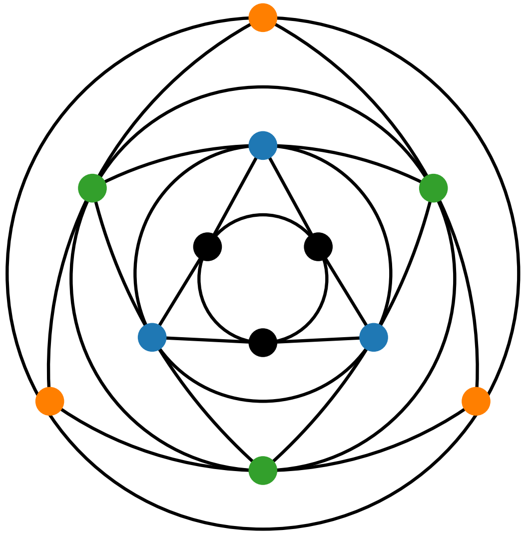

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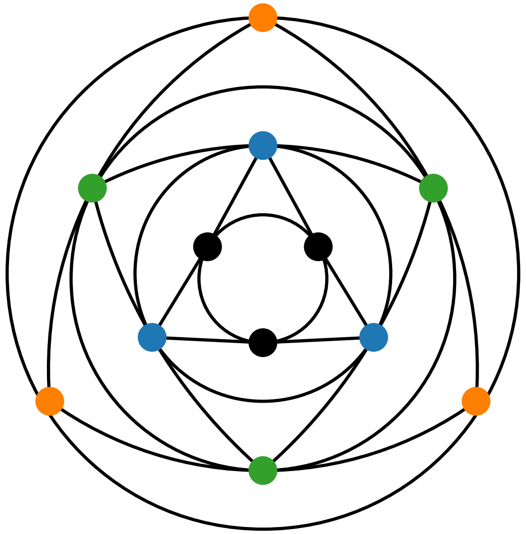

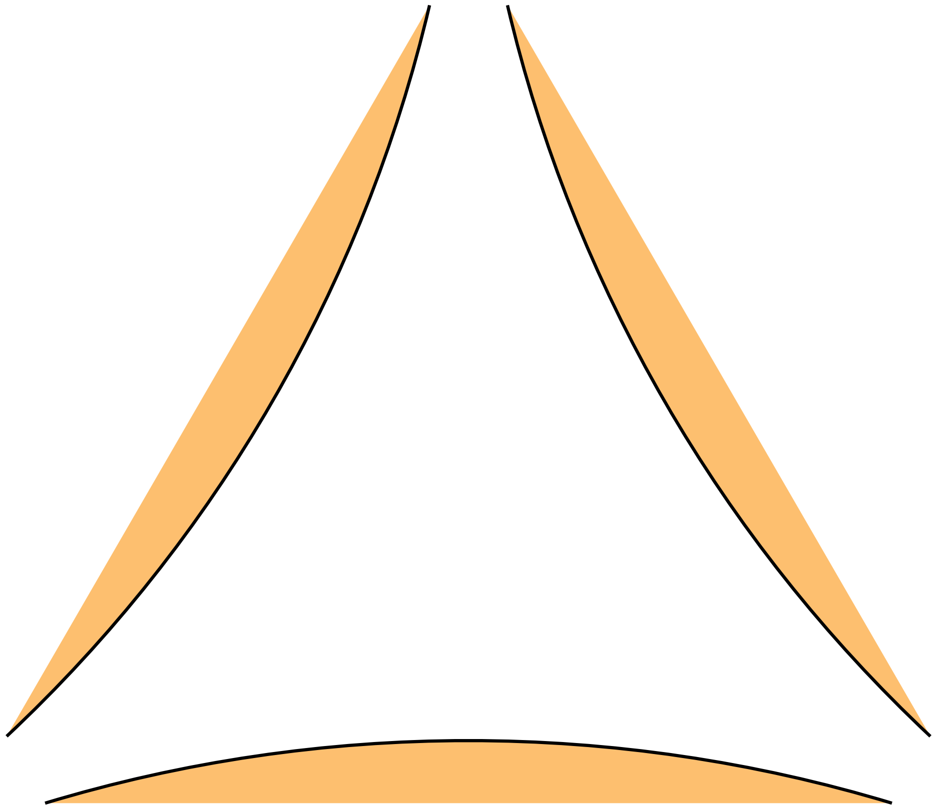

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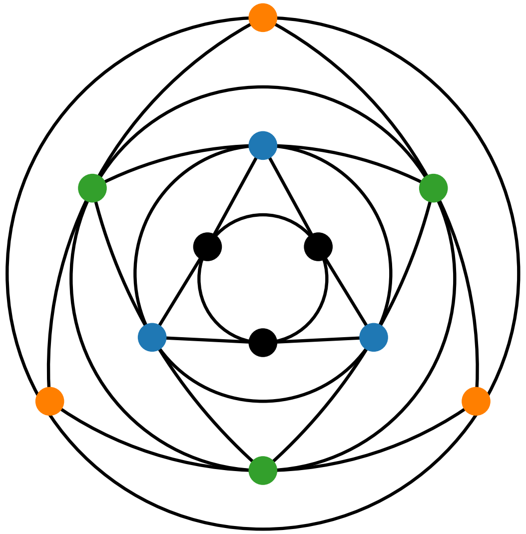

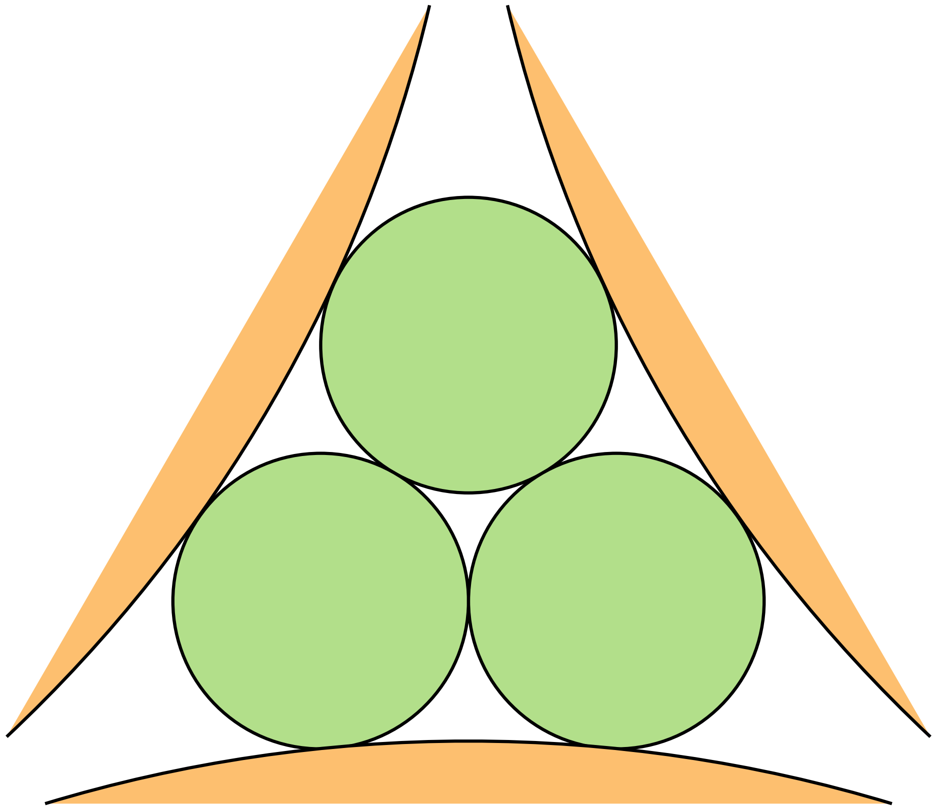

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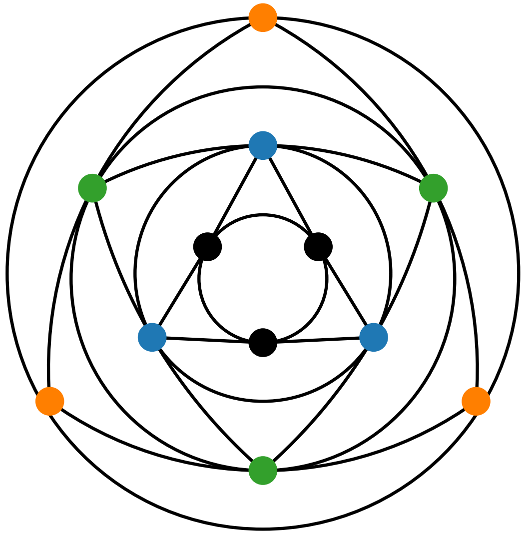

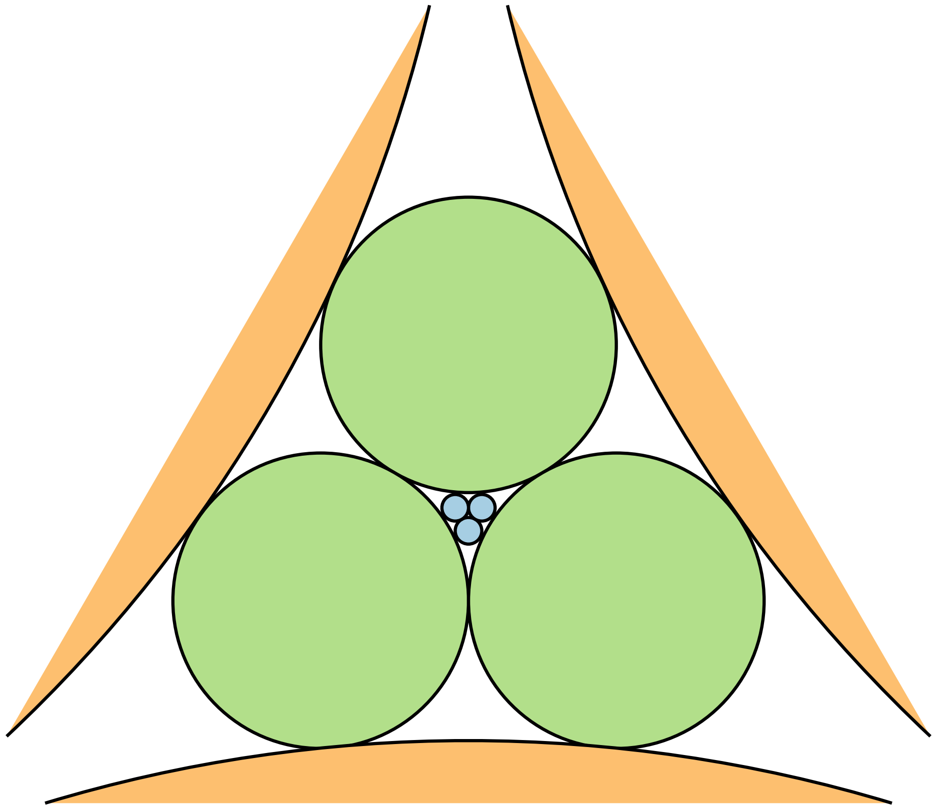

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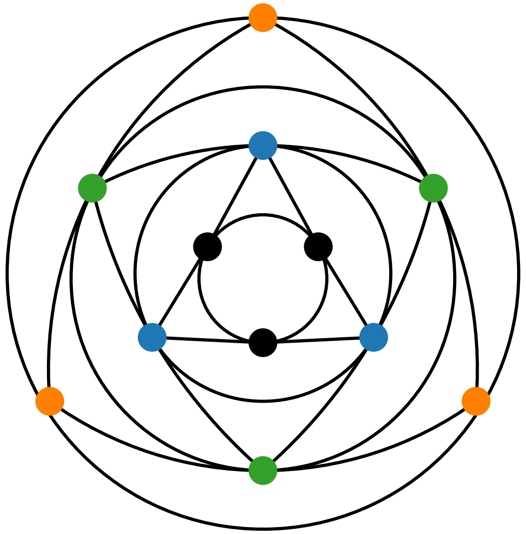

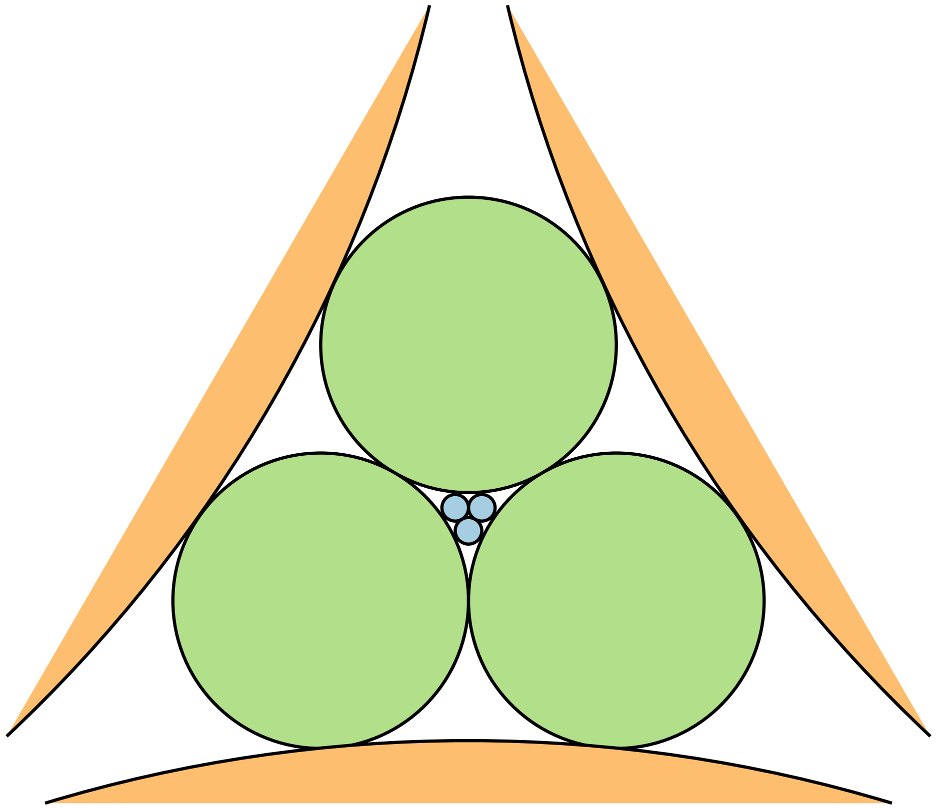

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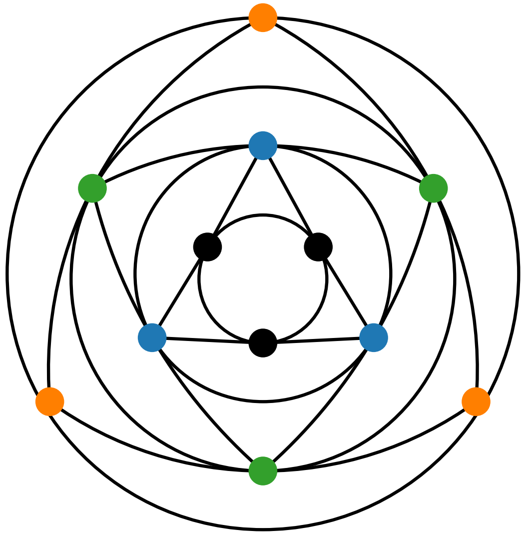

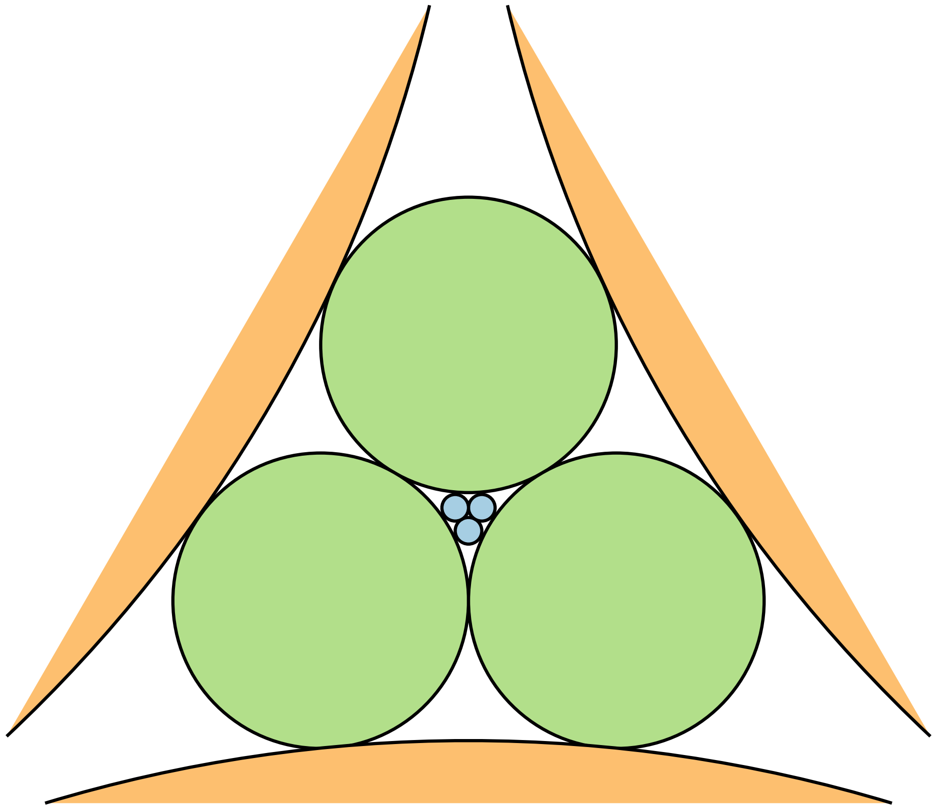

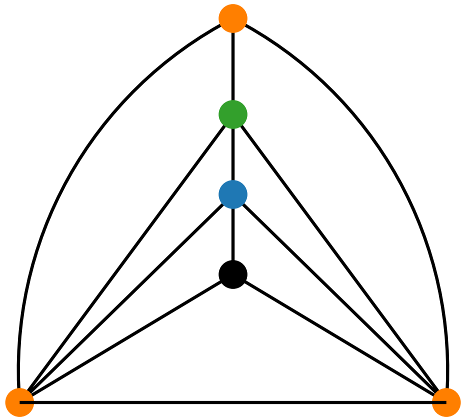

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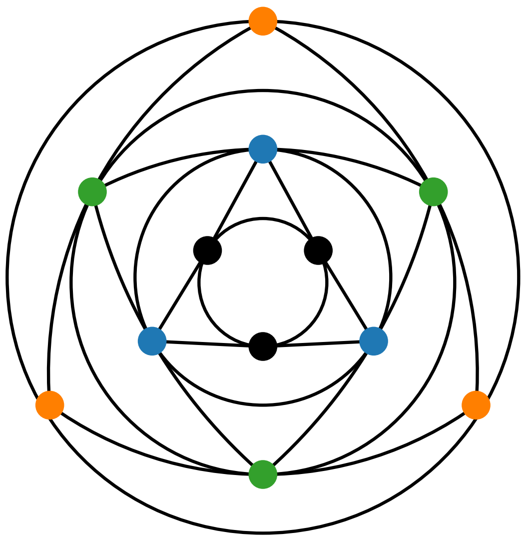

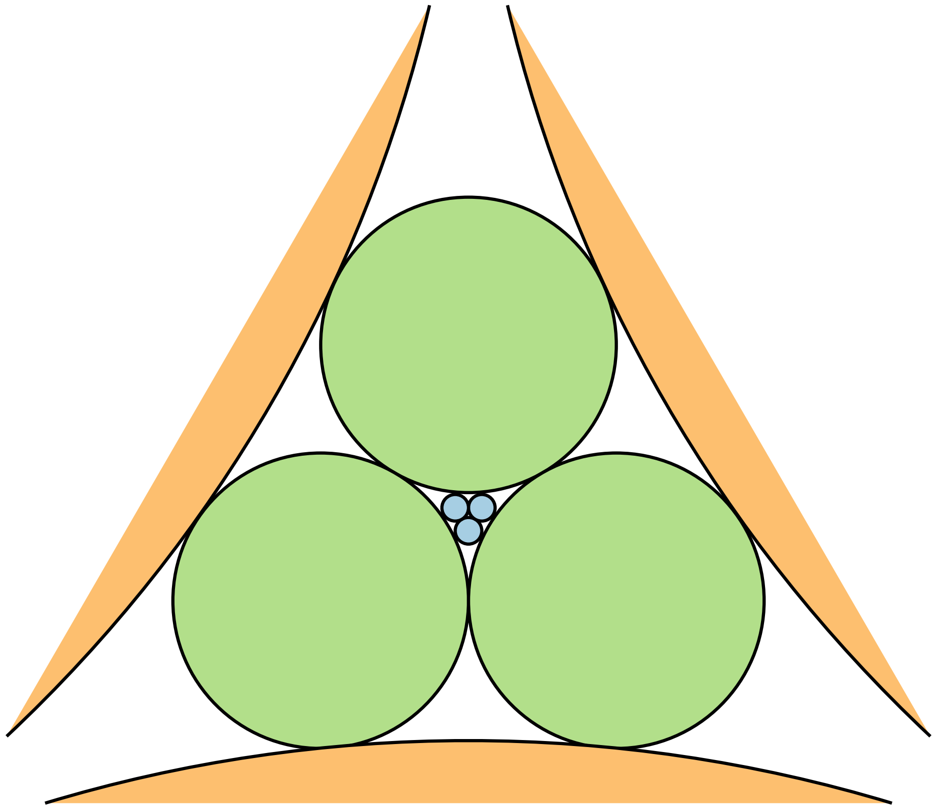

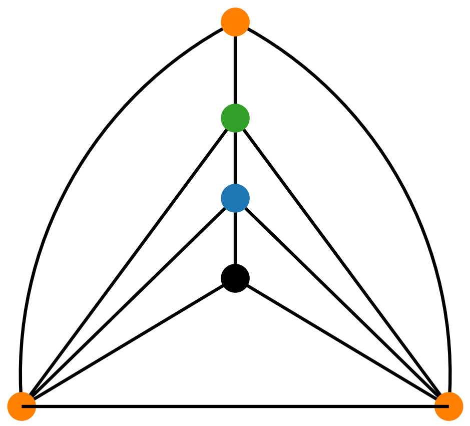

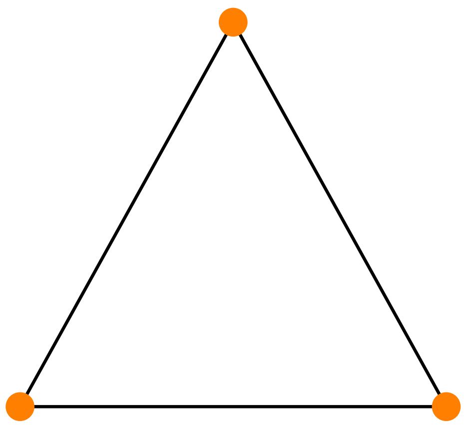

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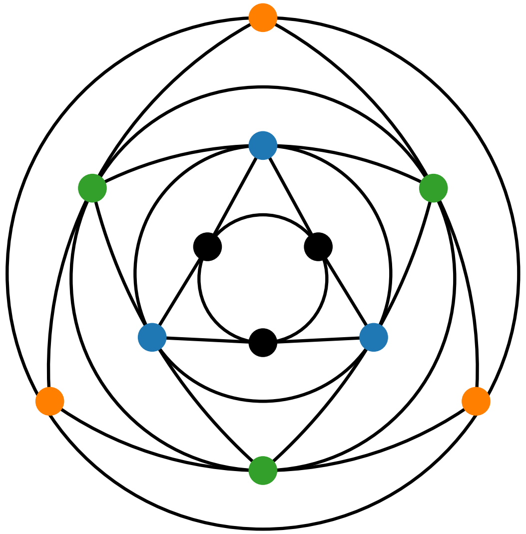

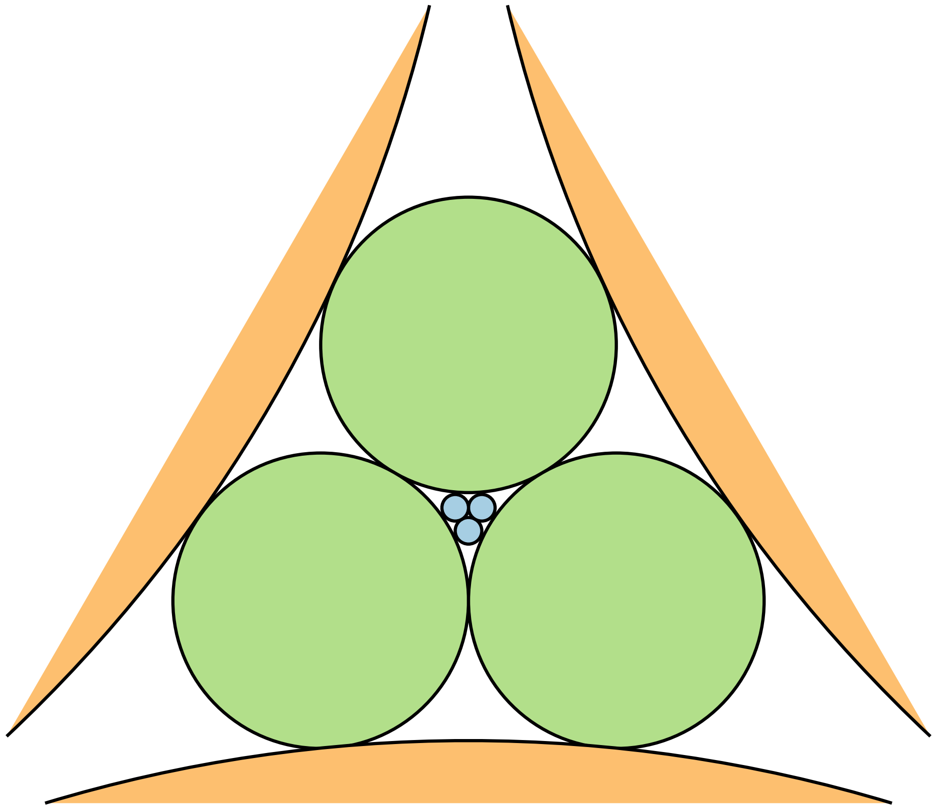

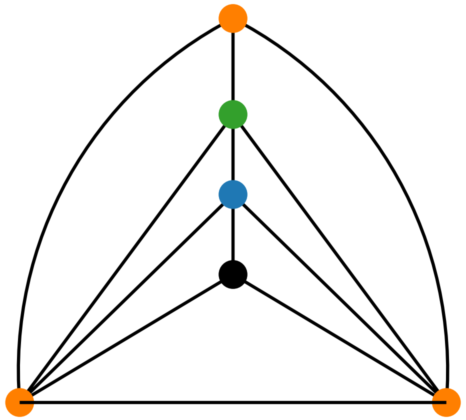

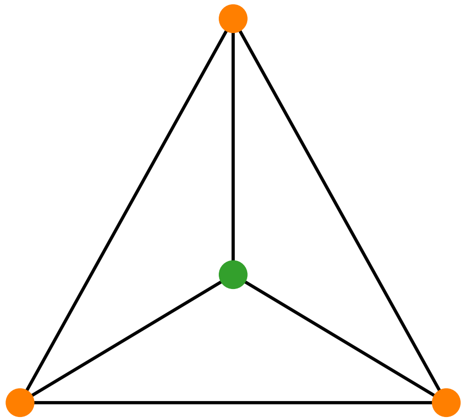

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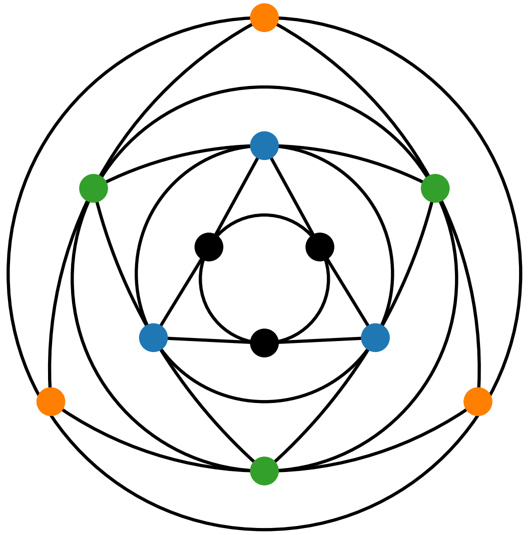

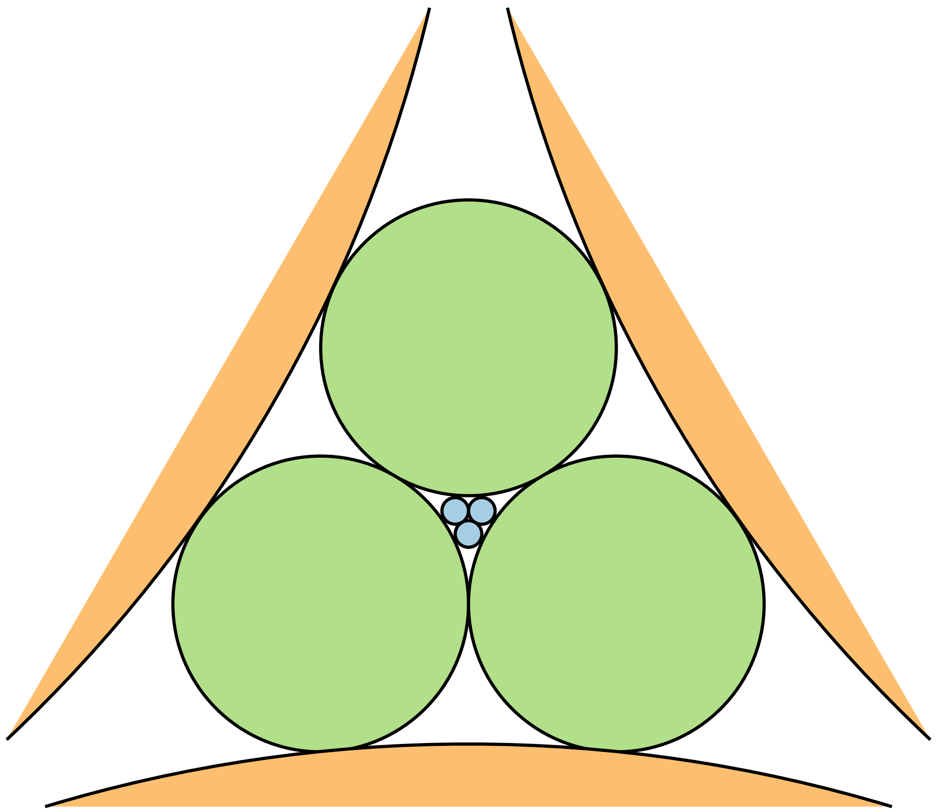

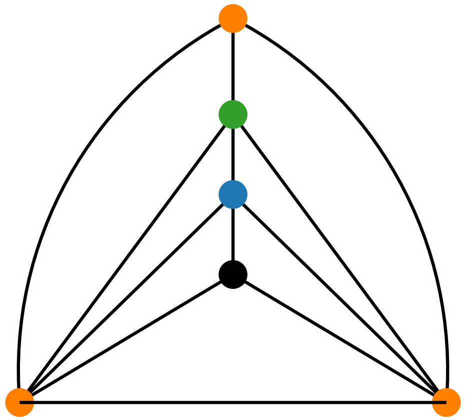

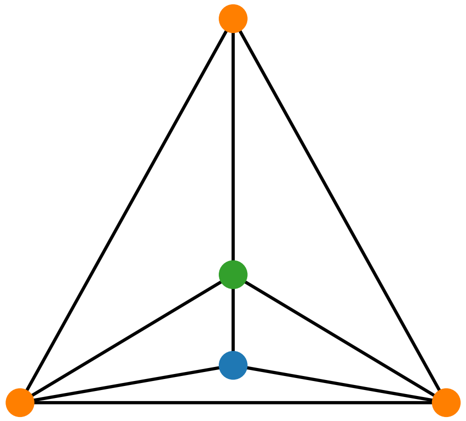

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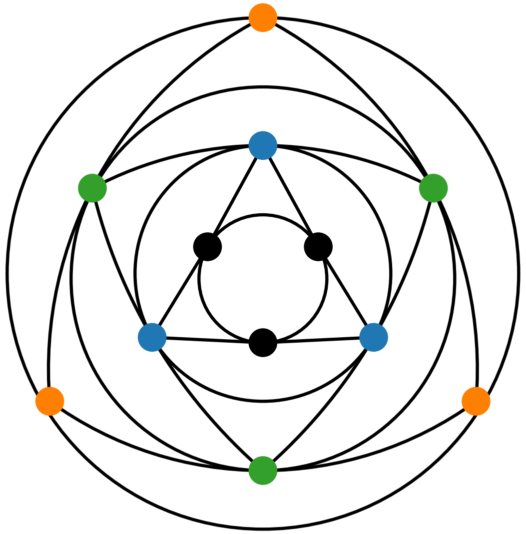

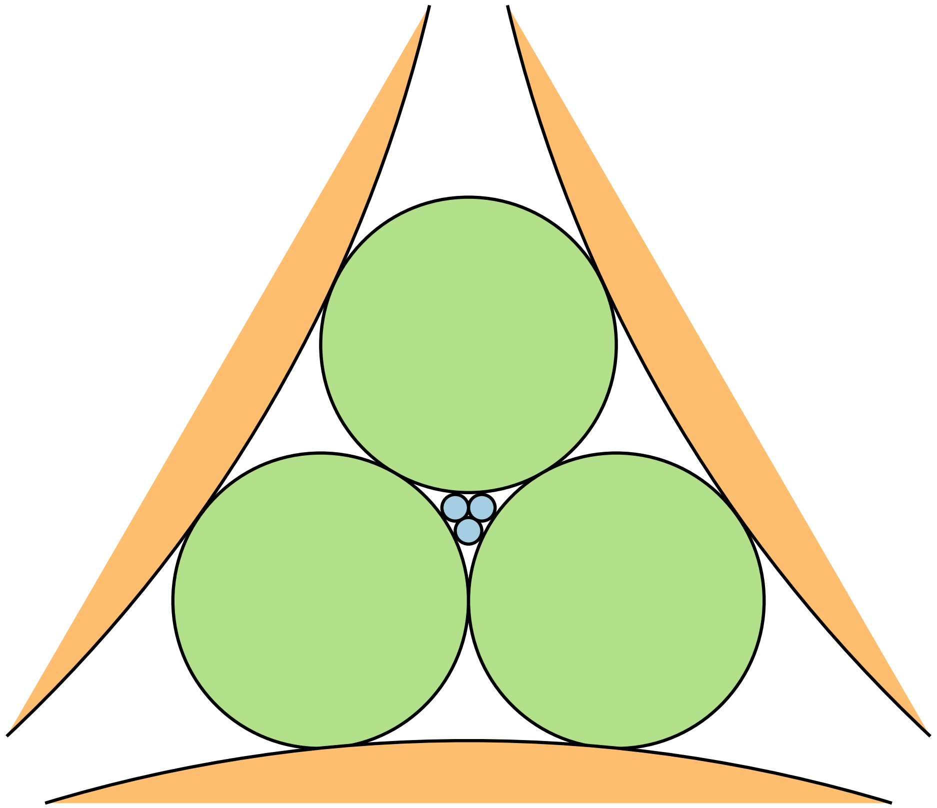

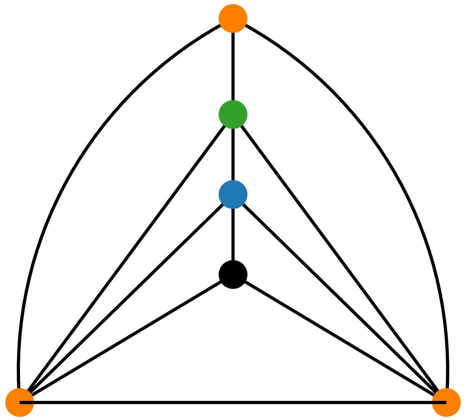

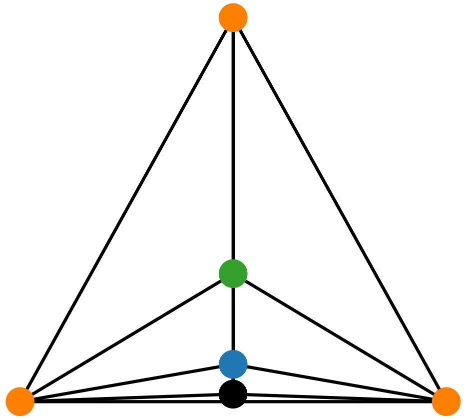

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- **Properties of planar triangulations:**
  - Every face is a triangle
  - graph is 3-connected
  - Unique embedding (up to choice of outerface)
  - Every plane graph is subgraph of a plane triangulation

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- A *planar triangulation* is a planar graph with $3n - 6$ edges

- **Properties of planar triangulations:**
  - Every face is a triangle
  - graph is 3-connected
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  - Every plane graph is subgraph of a plane triangulation

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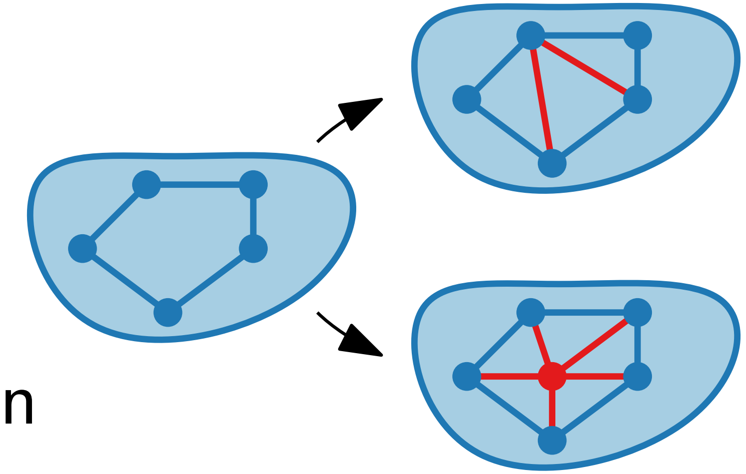

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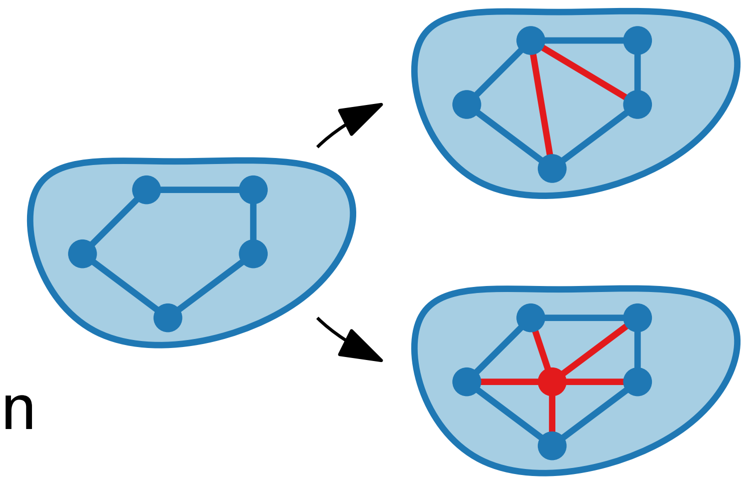
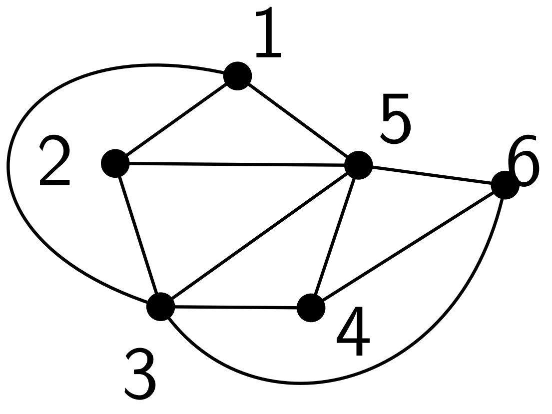


with planar embedding

- We focus on **triangulations**:
  - A *plane (inner) triangulation* is a plane graph where every (inner) face is a triangle.



Planar straight-line drawings

Goal:
For an n-vertex planar graph create a planar straight-line drawing of size $\mathcal{O}(n^2)$.

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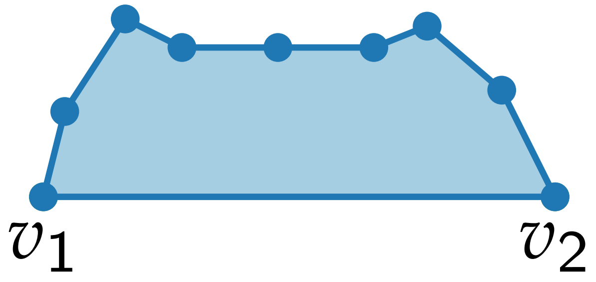

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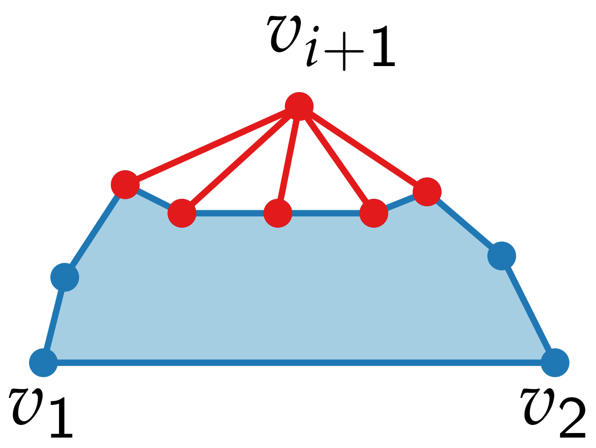
- To obtain G_{i+1}, add v_{i+1} to G_i so that neighbours of v_{i+1} are on the outer face of G_i.
- Neighbours of v_{i+1} in G_i have to form path of length at least two.



Canonical order – definition

Definition.
Let $G = (V, E)$ be a triangulated plane graph on $n \geq 3$ vertices.
An order $\pi = (v_1, v_2, \ldots, v_n)$ is called a **canonical order**, if the following conditions hold for each k, $3 \leq k \leq n$:

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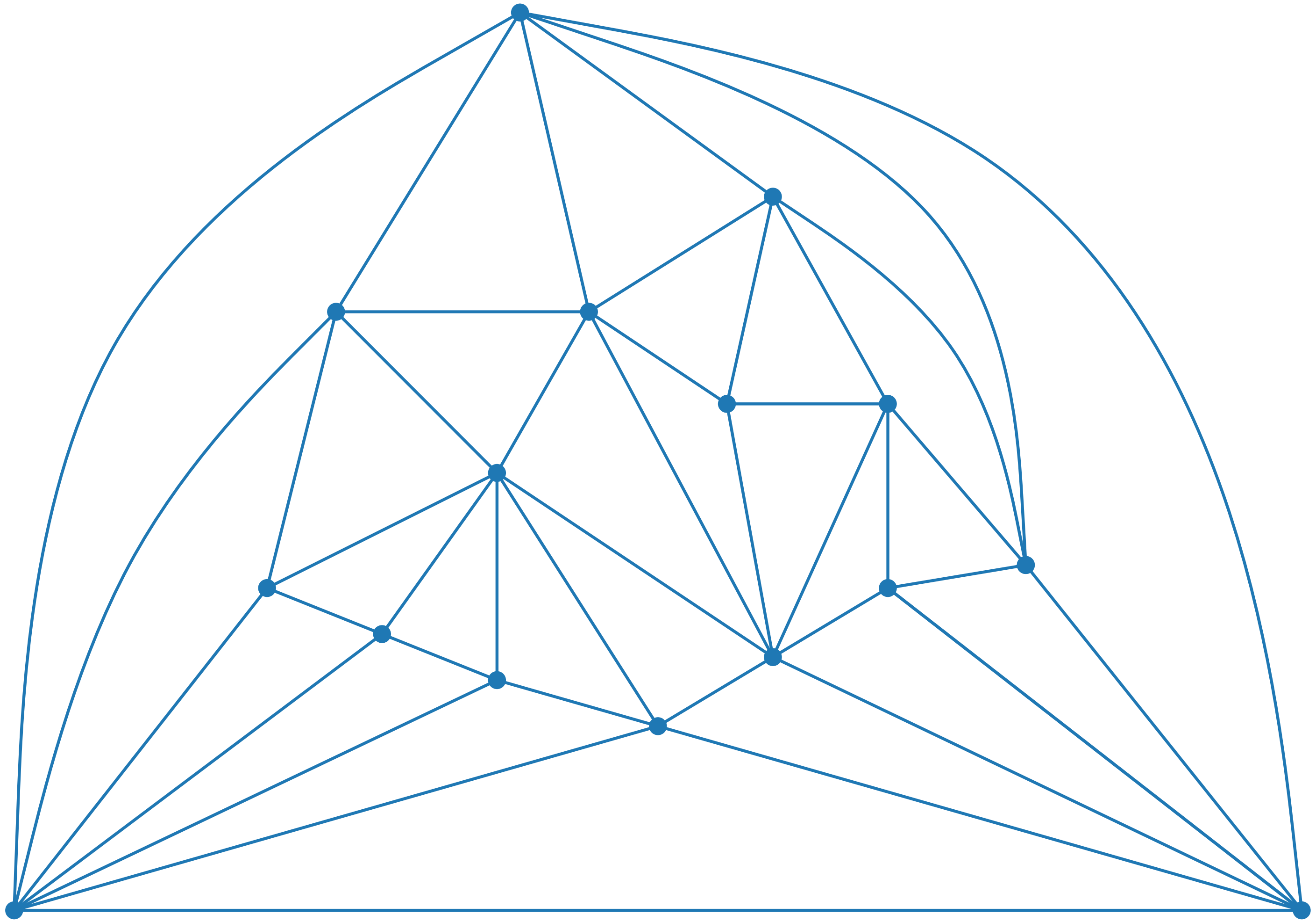
Compute:

- either $\{v_3, v_4, \ldots v_n\}$ (adding vertices)
- or $\{v_n, v_{n-1}, \ldots v_3\}$ (removing vertices)

Canonical order – example



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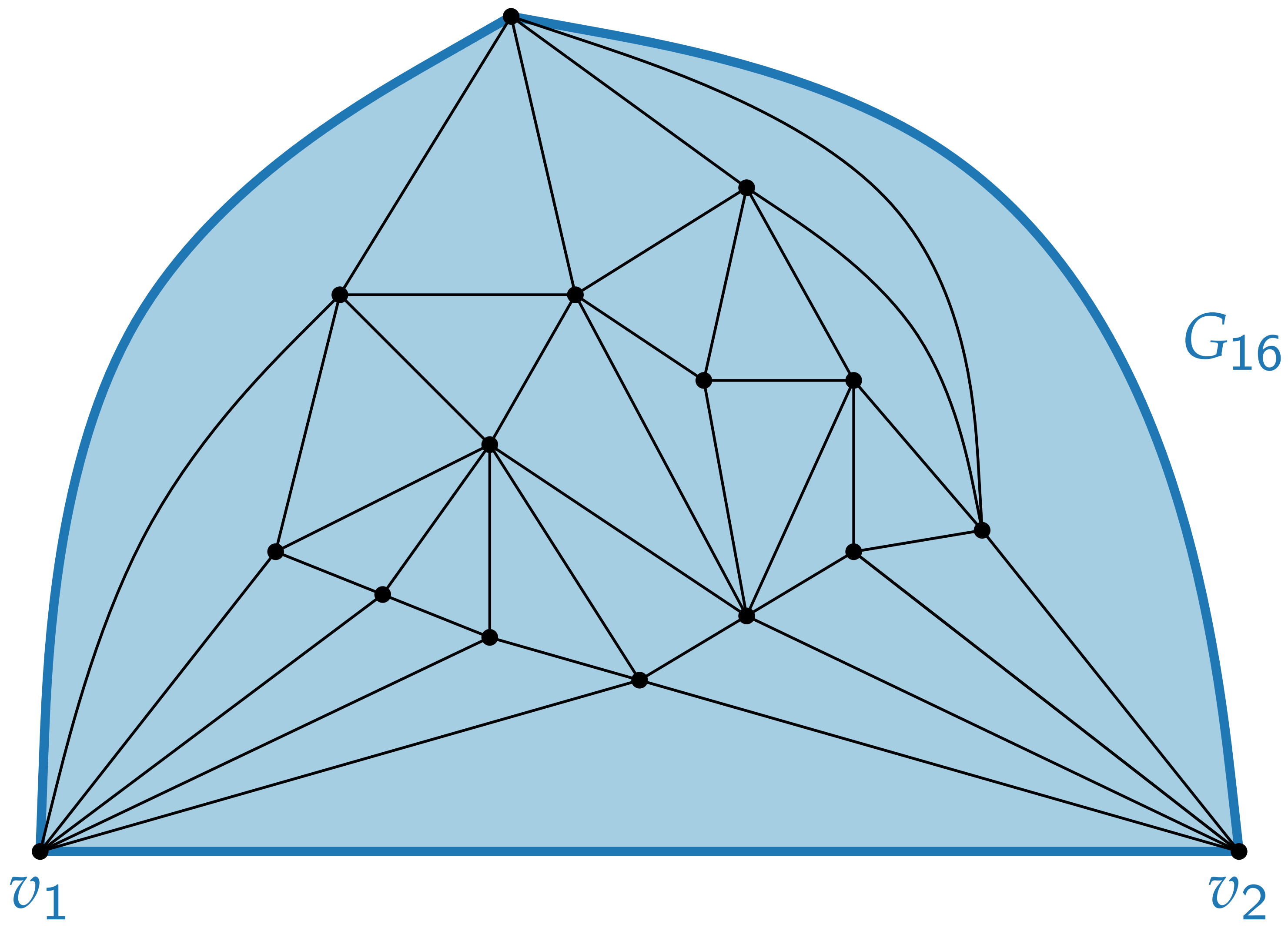

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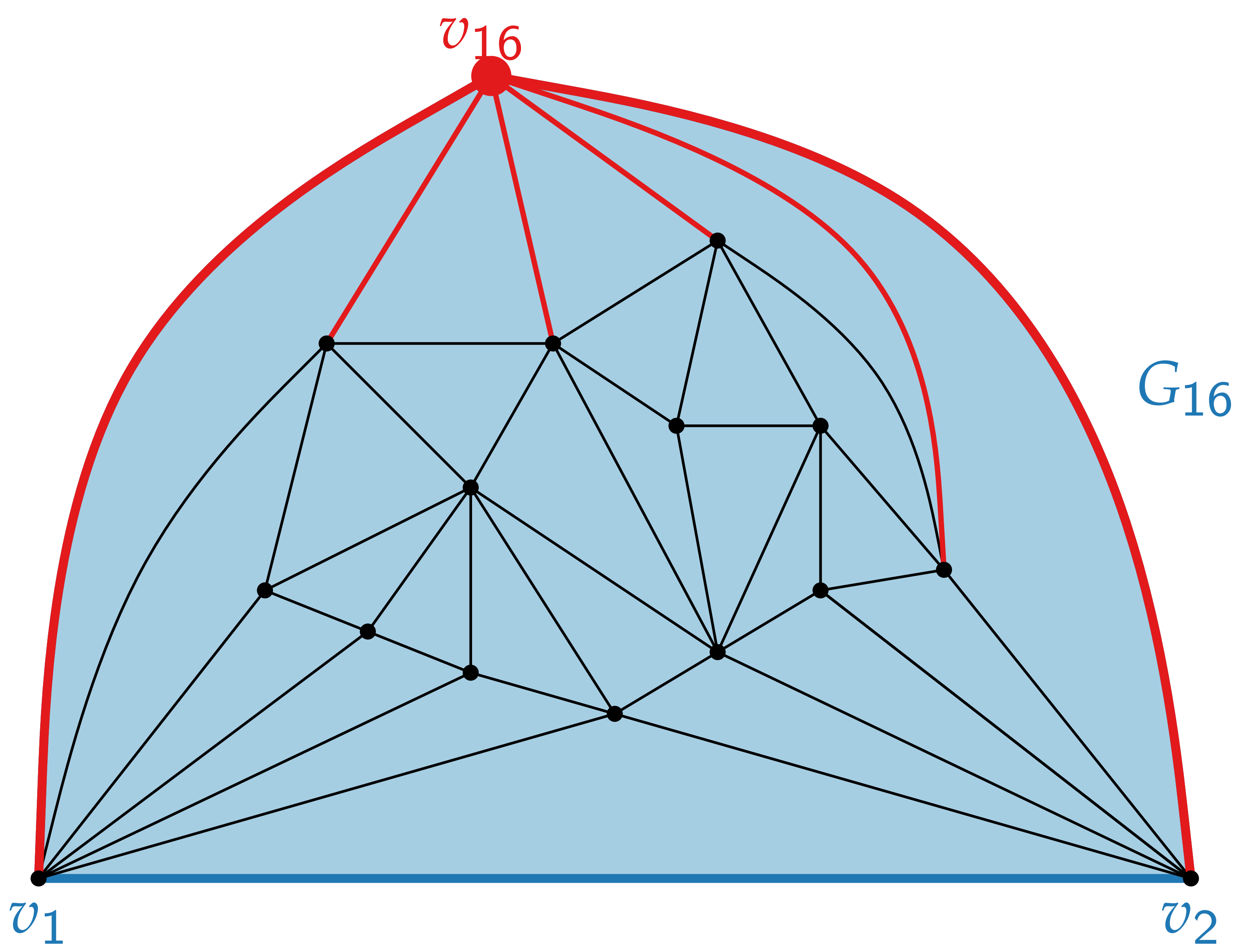

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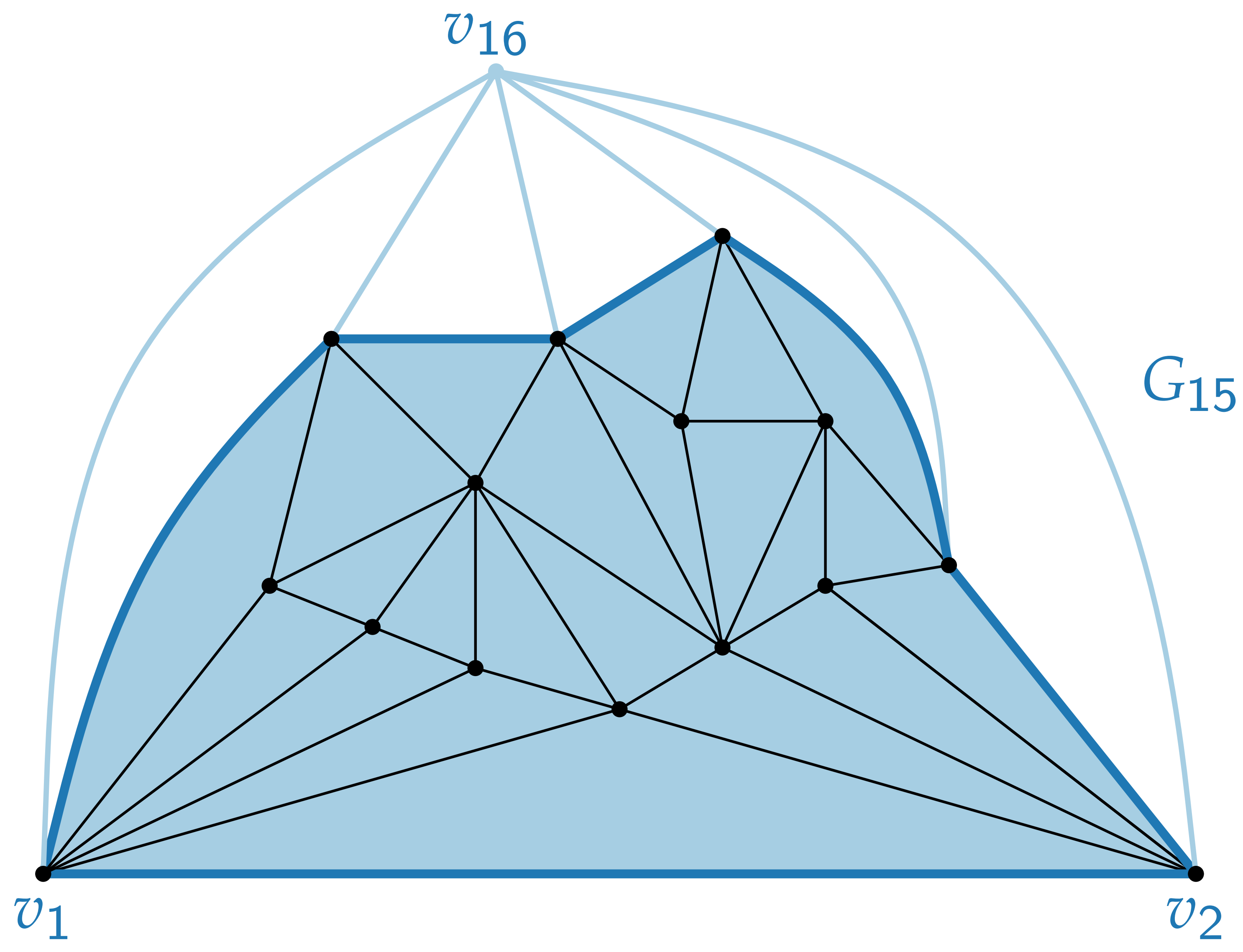

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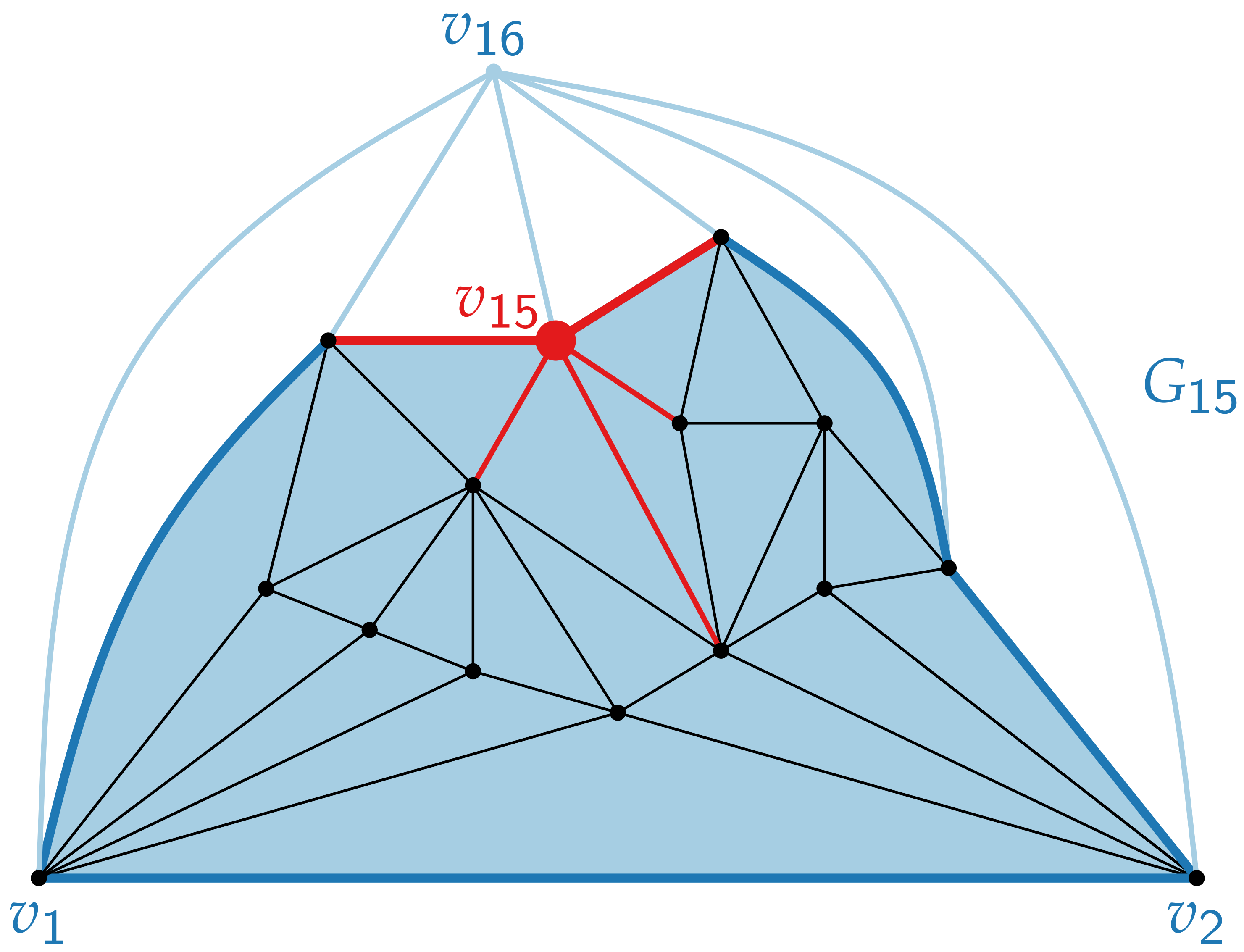

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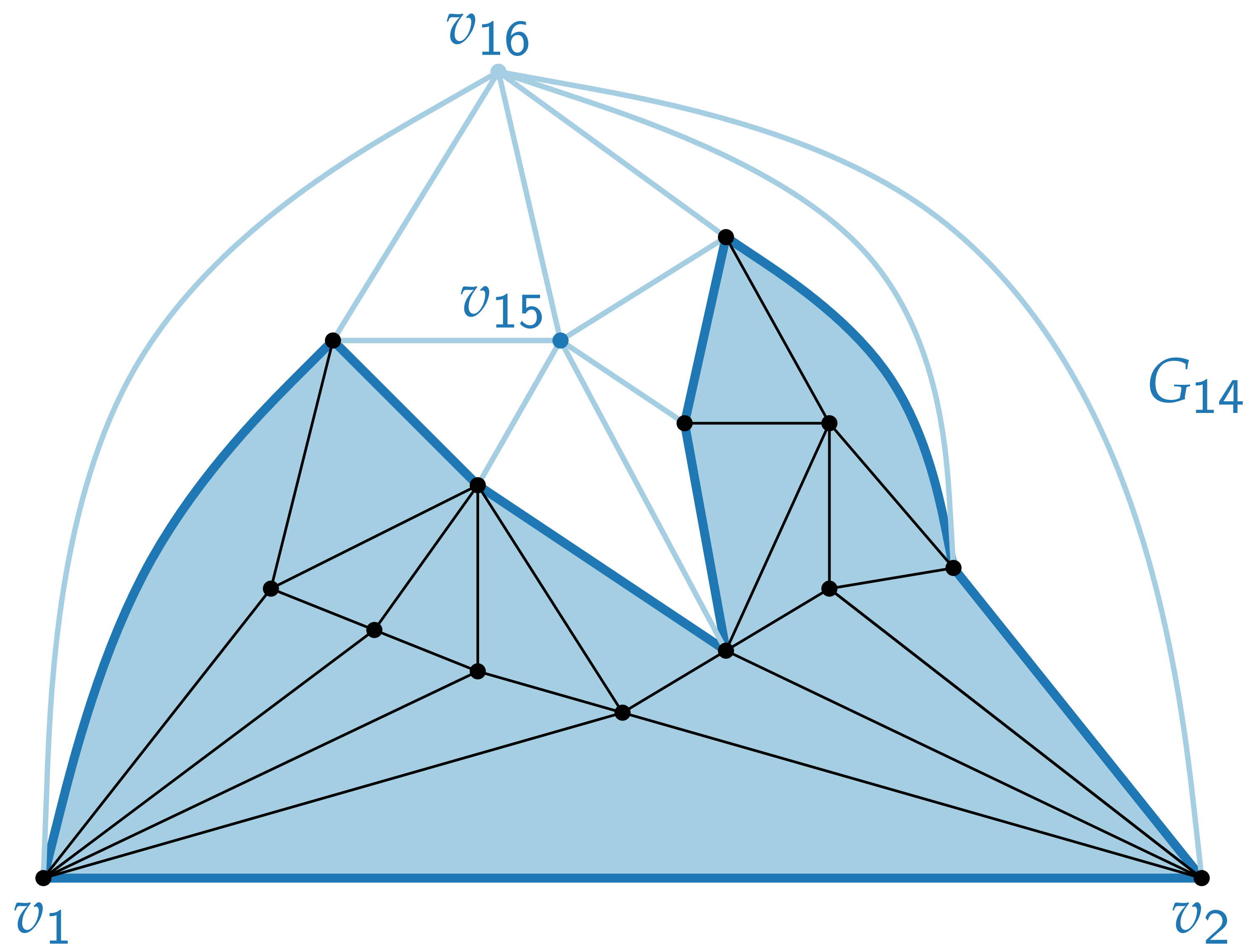

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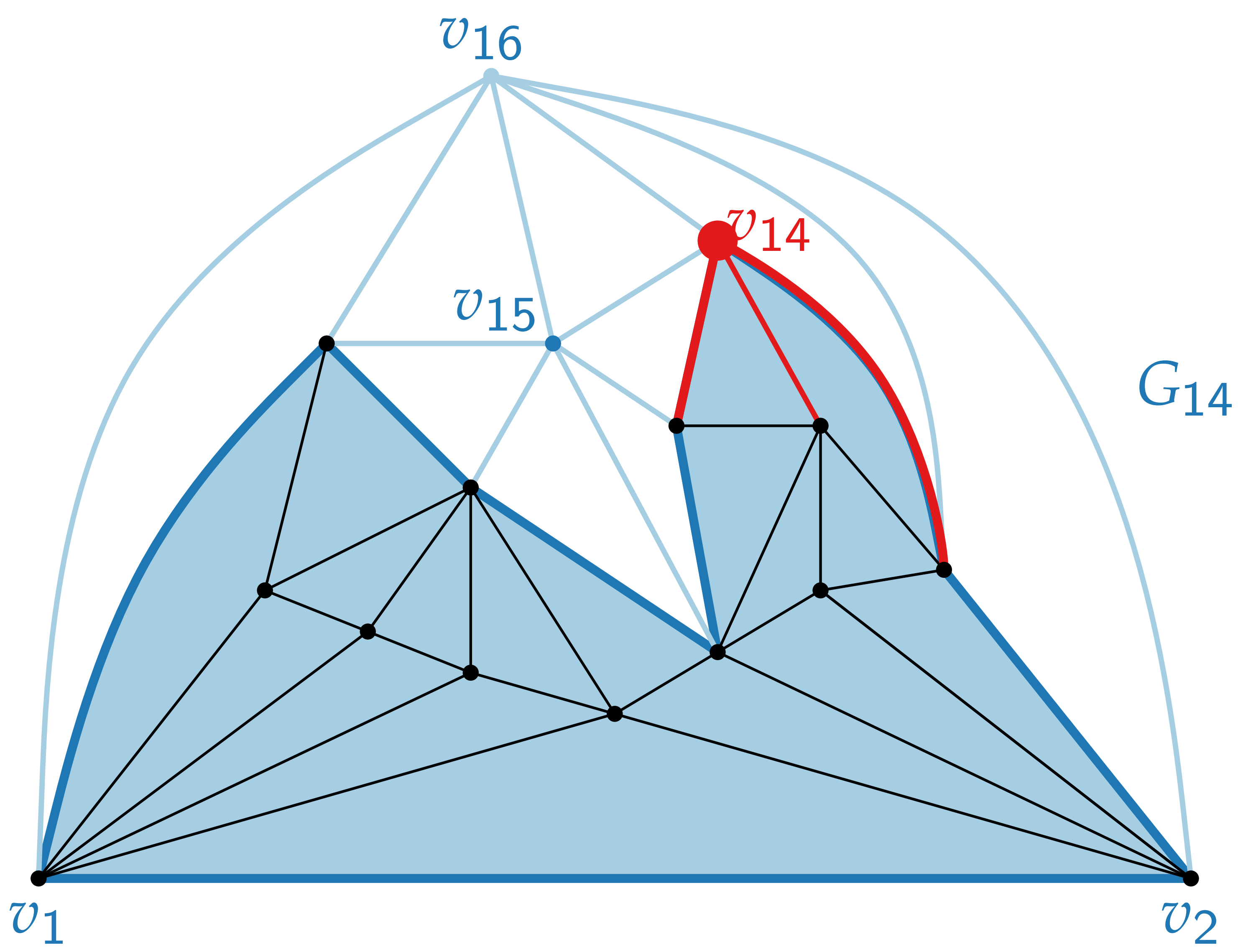

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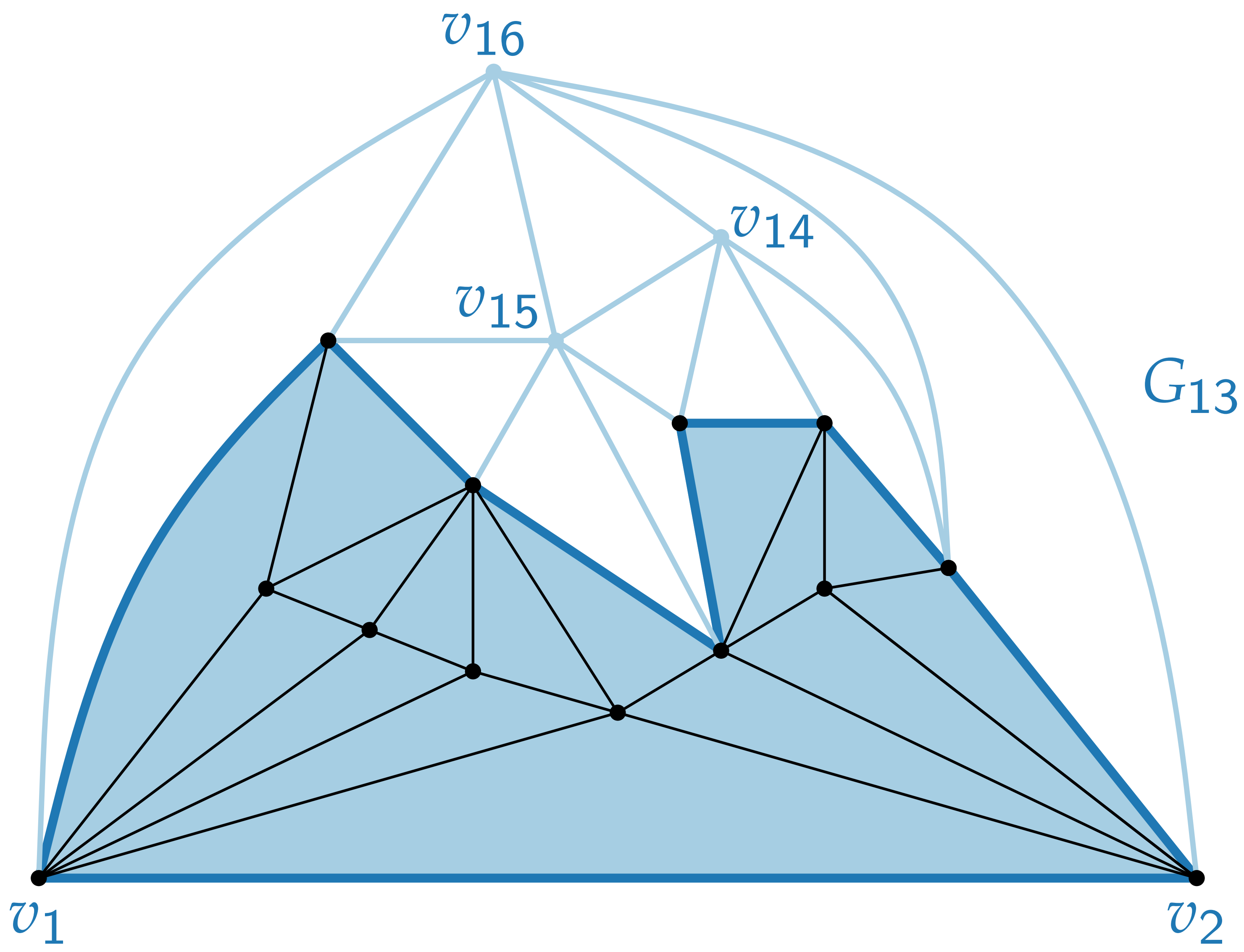

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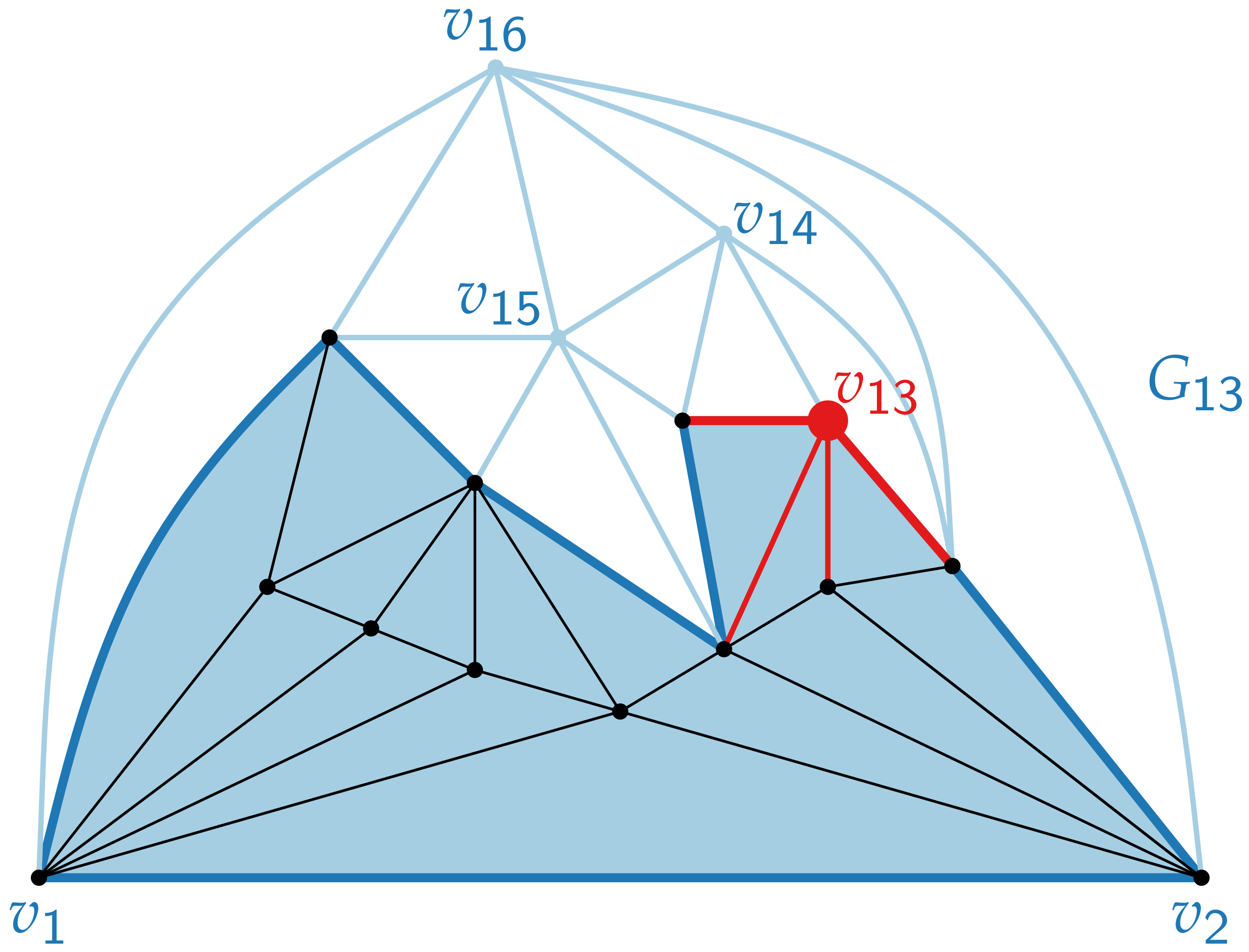

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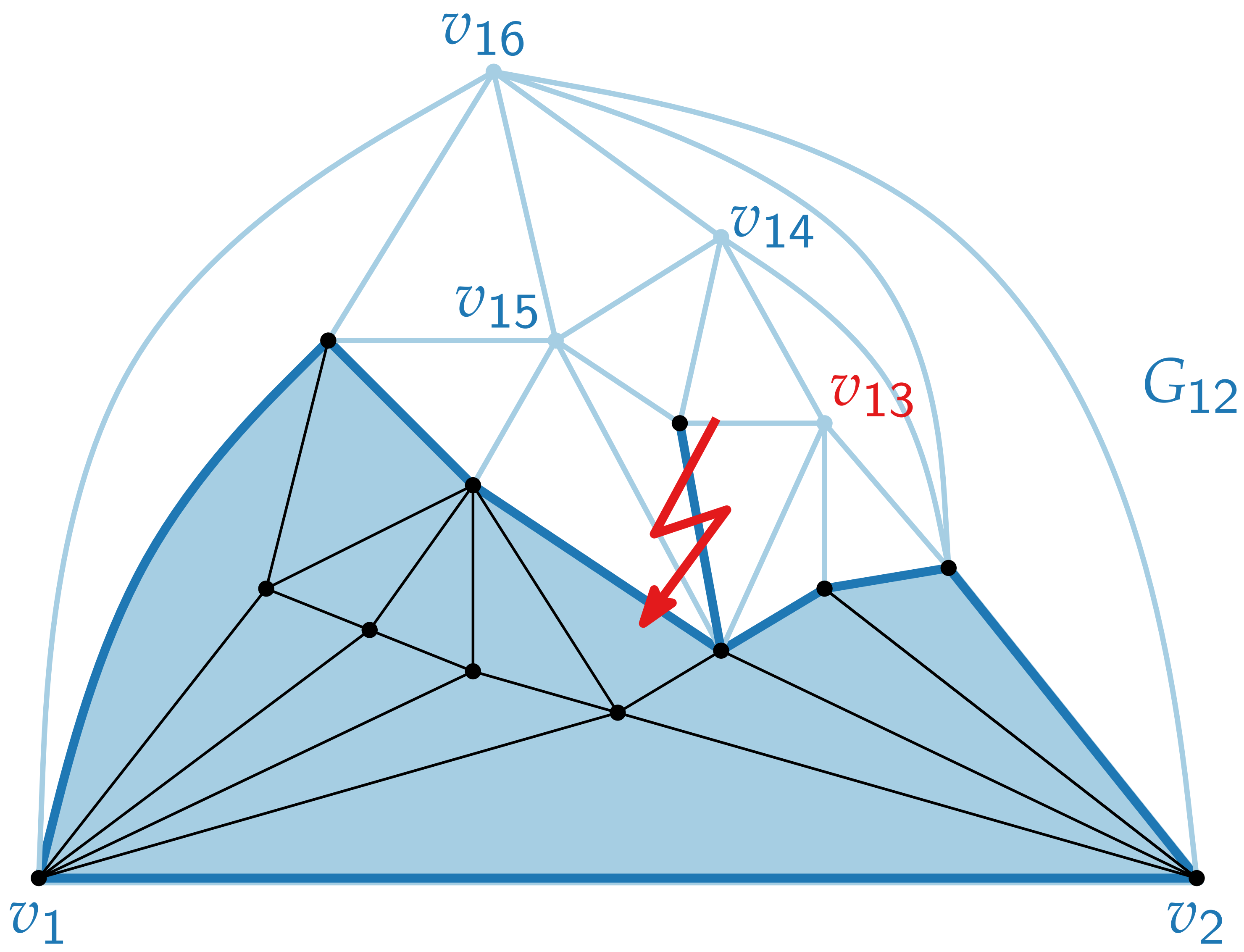

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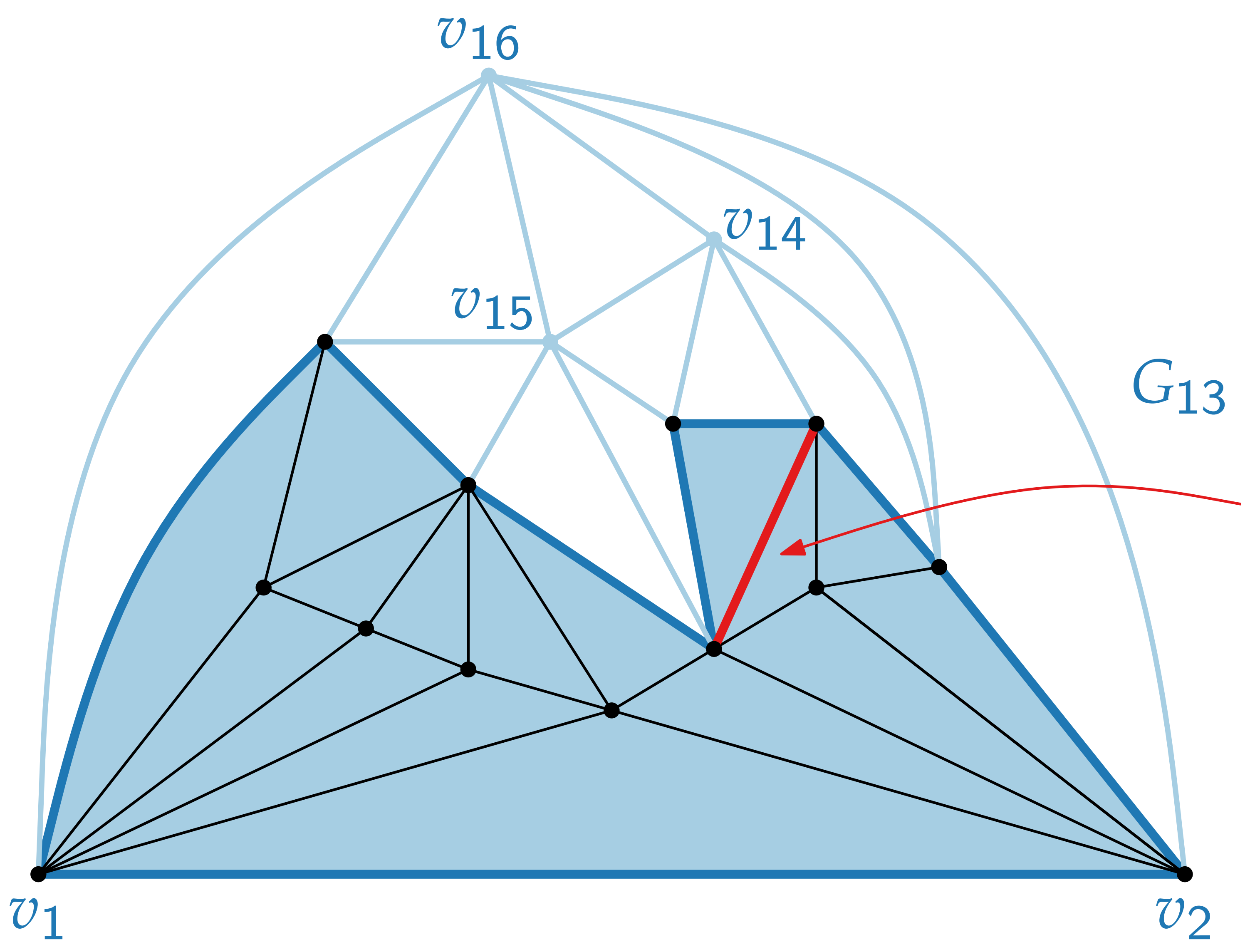
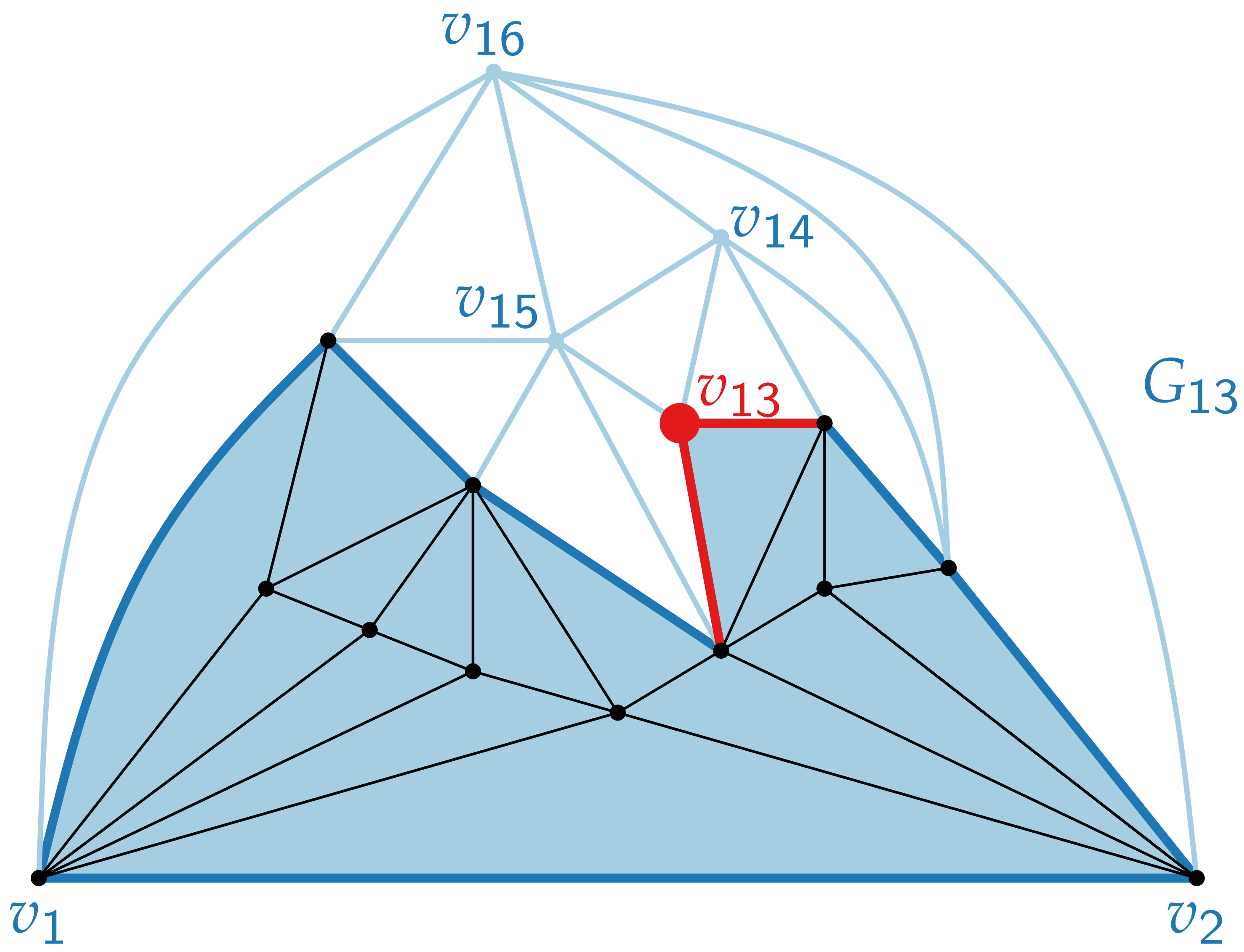


chord
edge joining two nonadjacent vertices in a cycle

Canonical order – example



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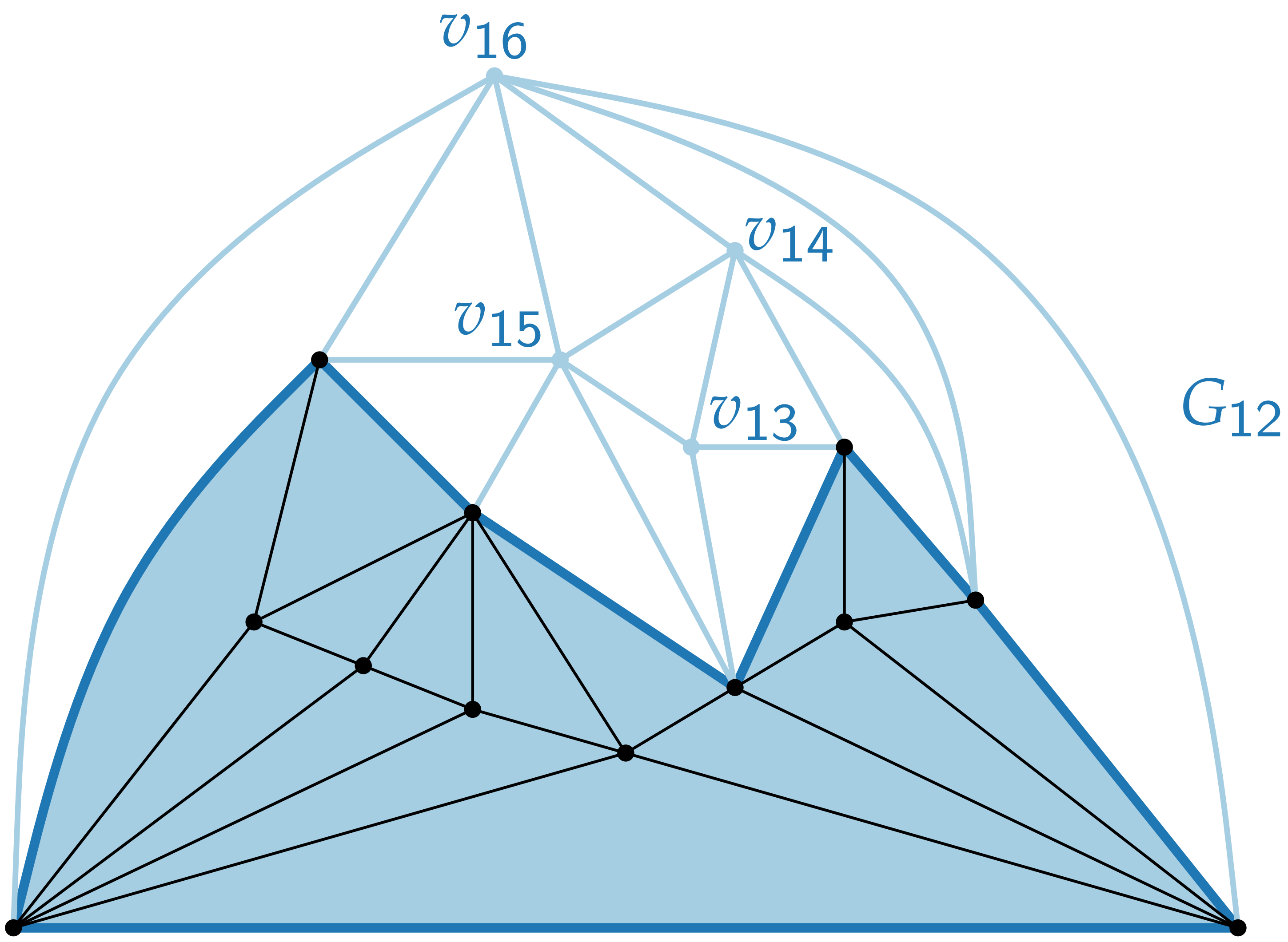

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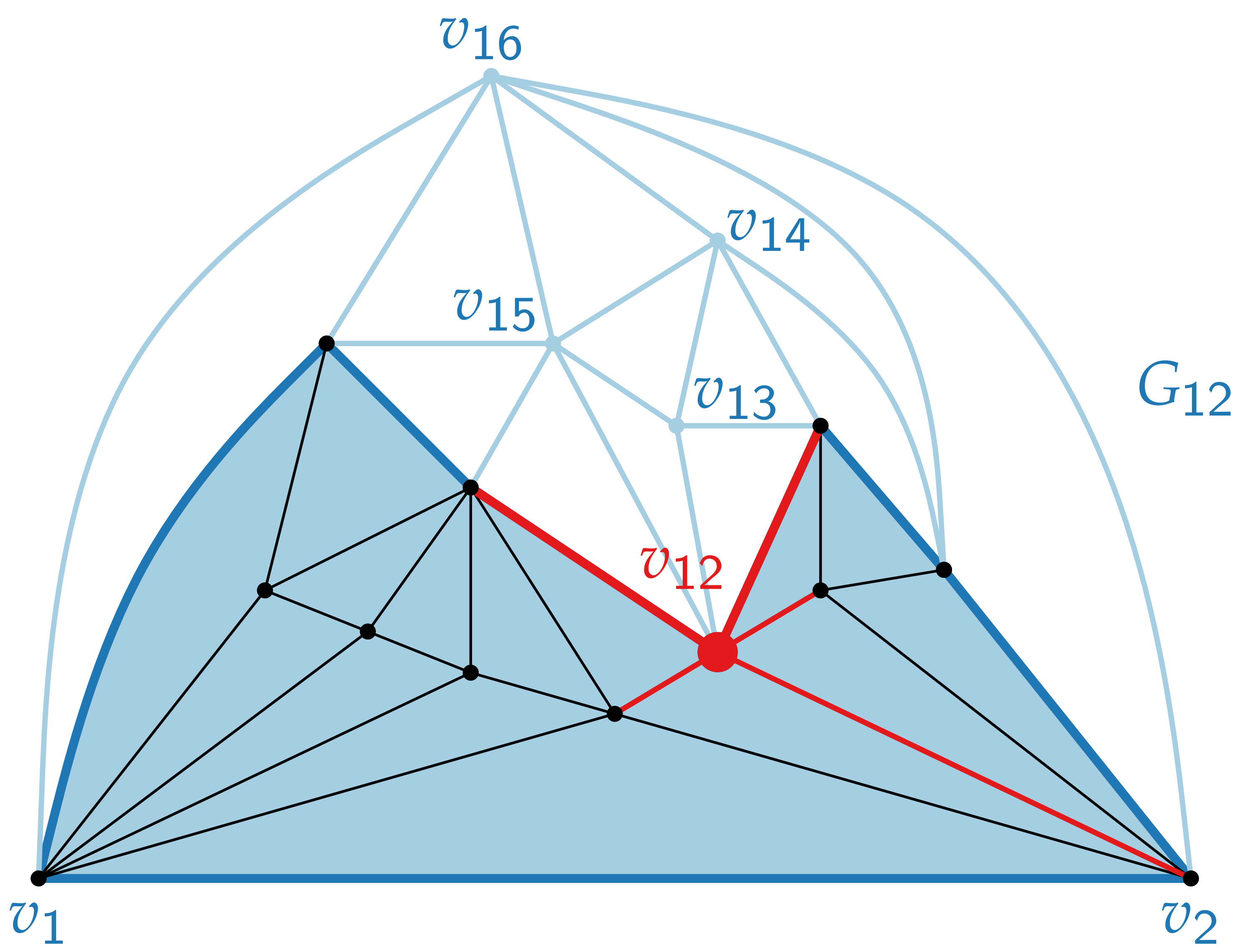

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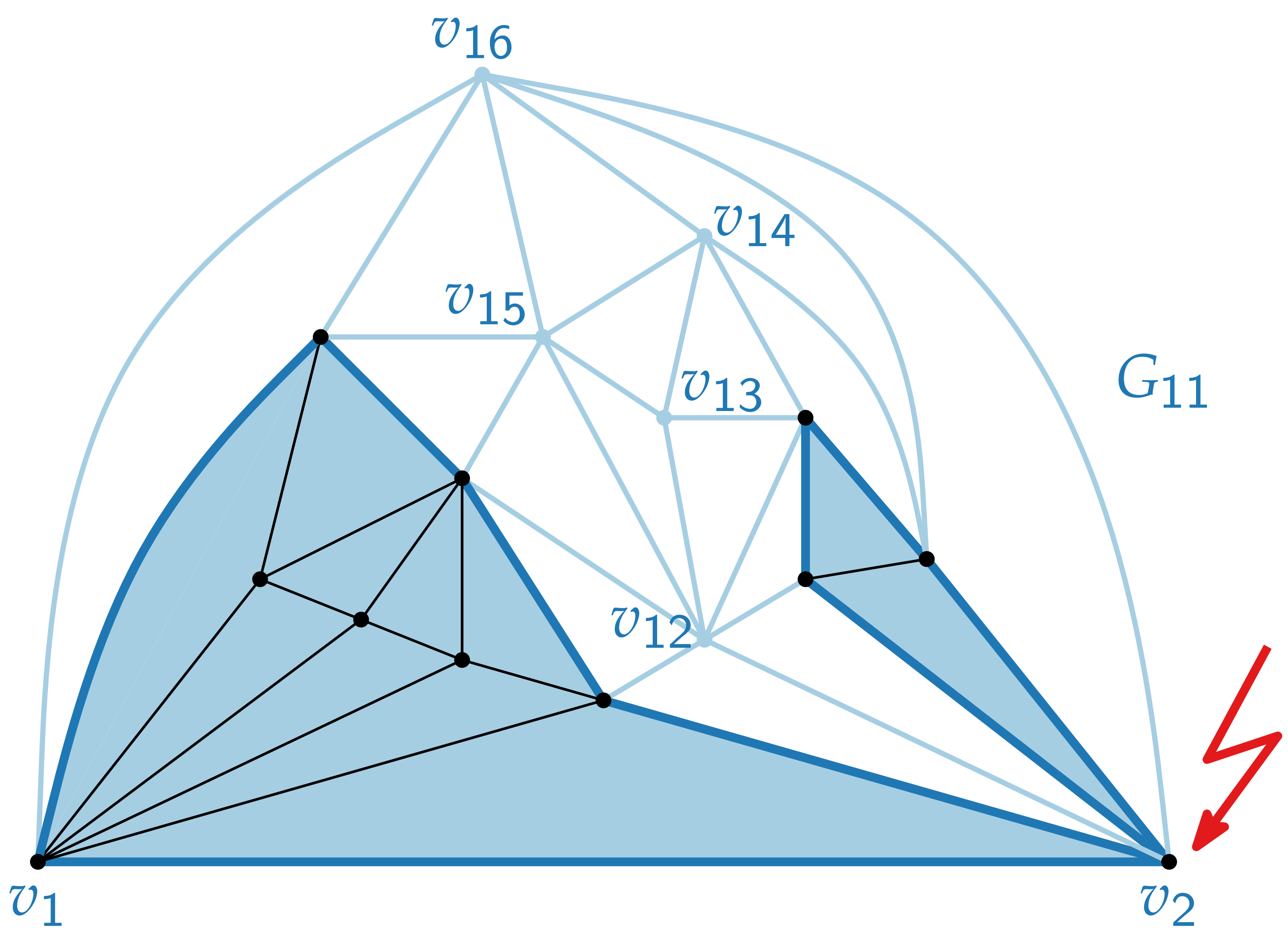

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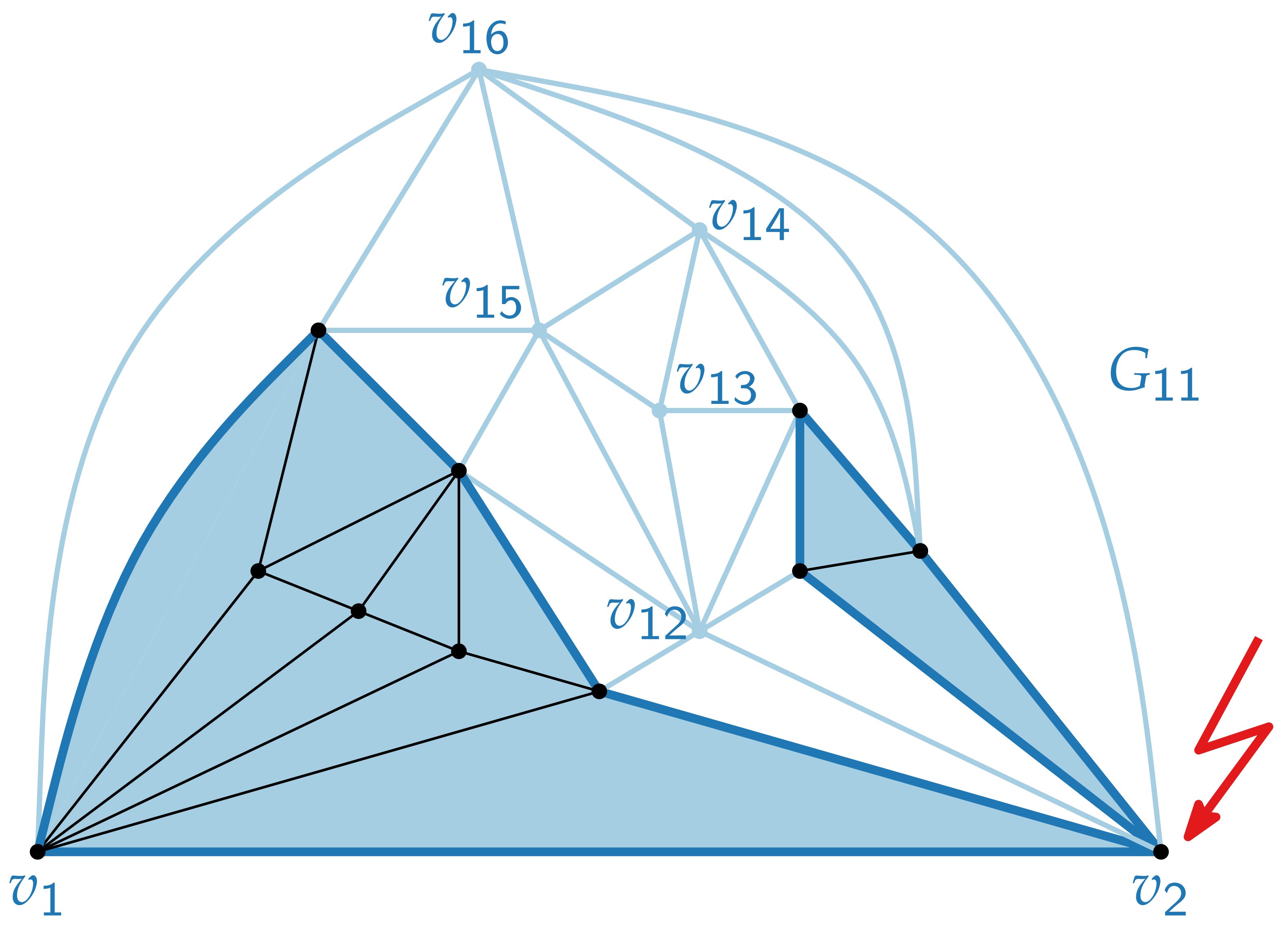
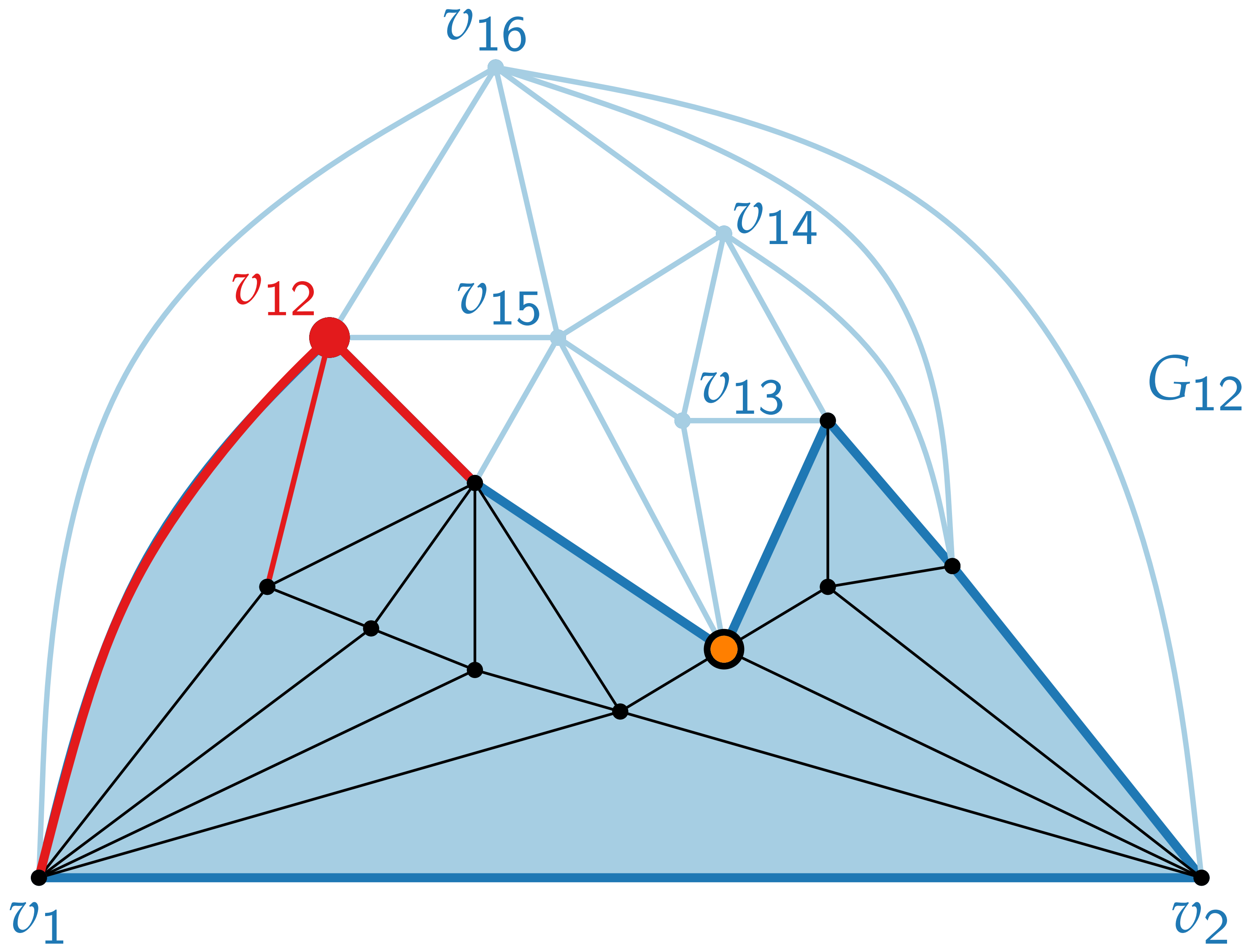


cutvertex

G_{11} is not biconnected

Canonical order – example



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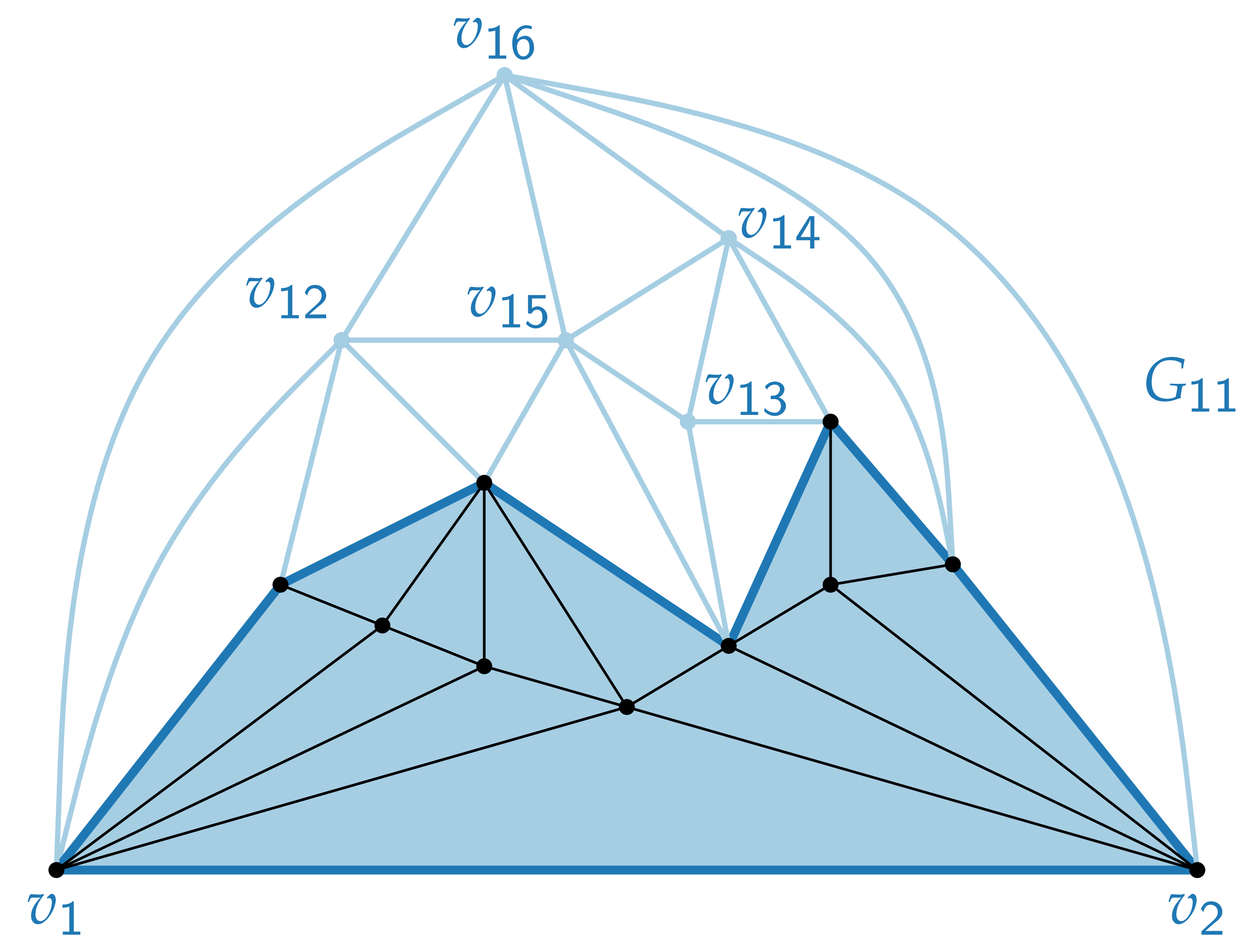

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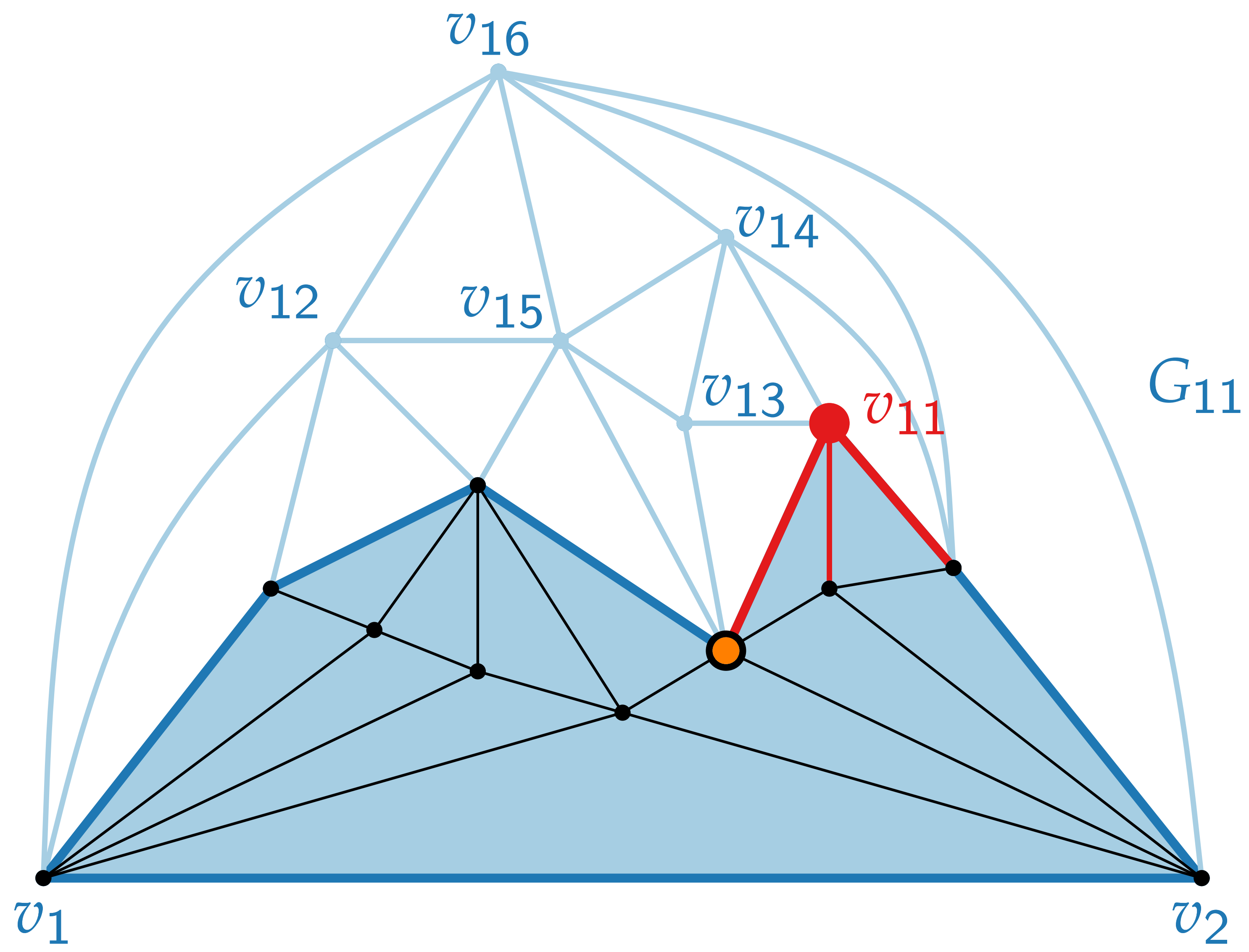

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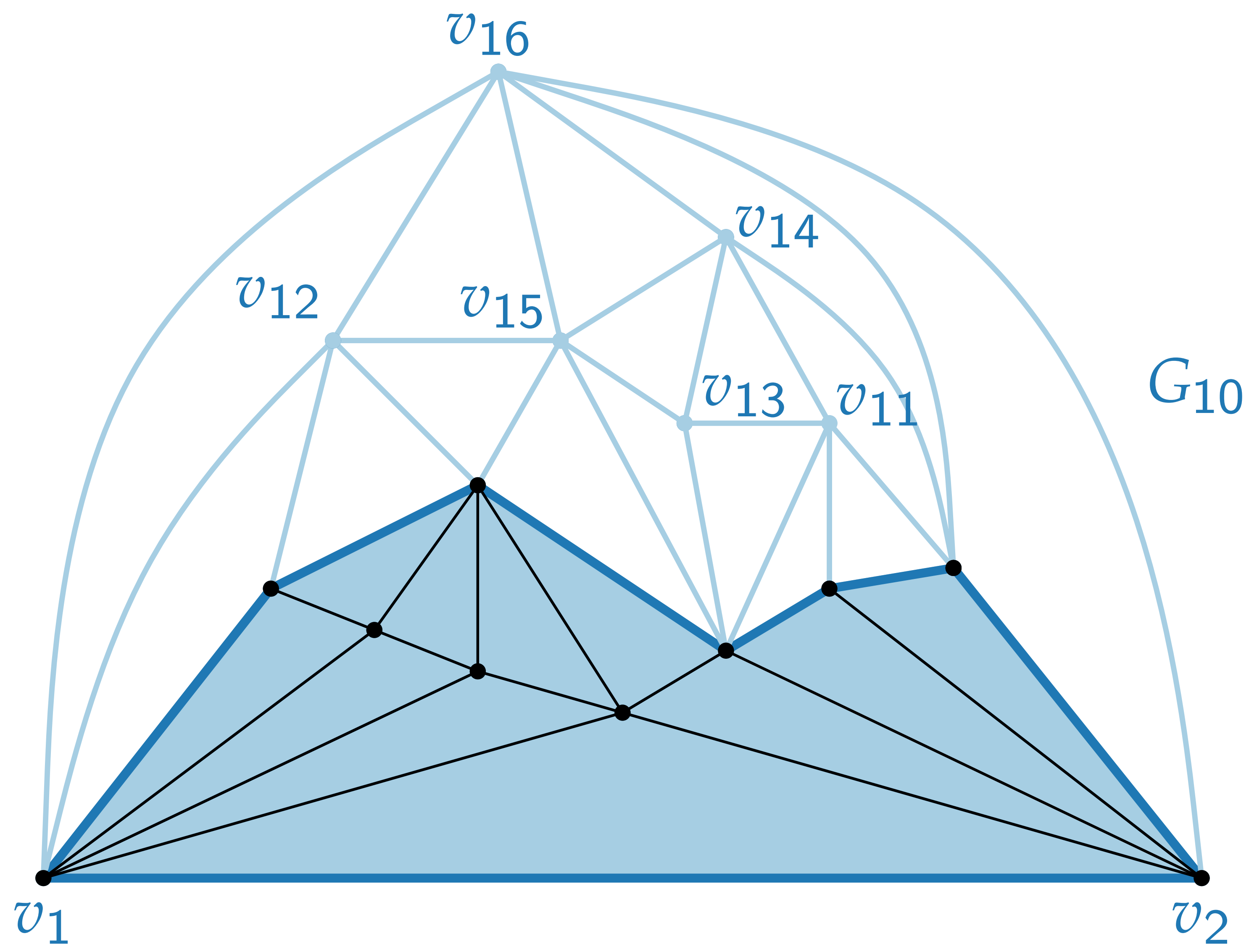

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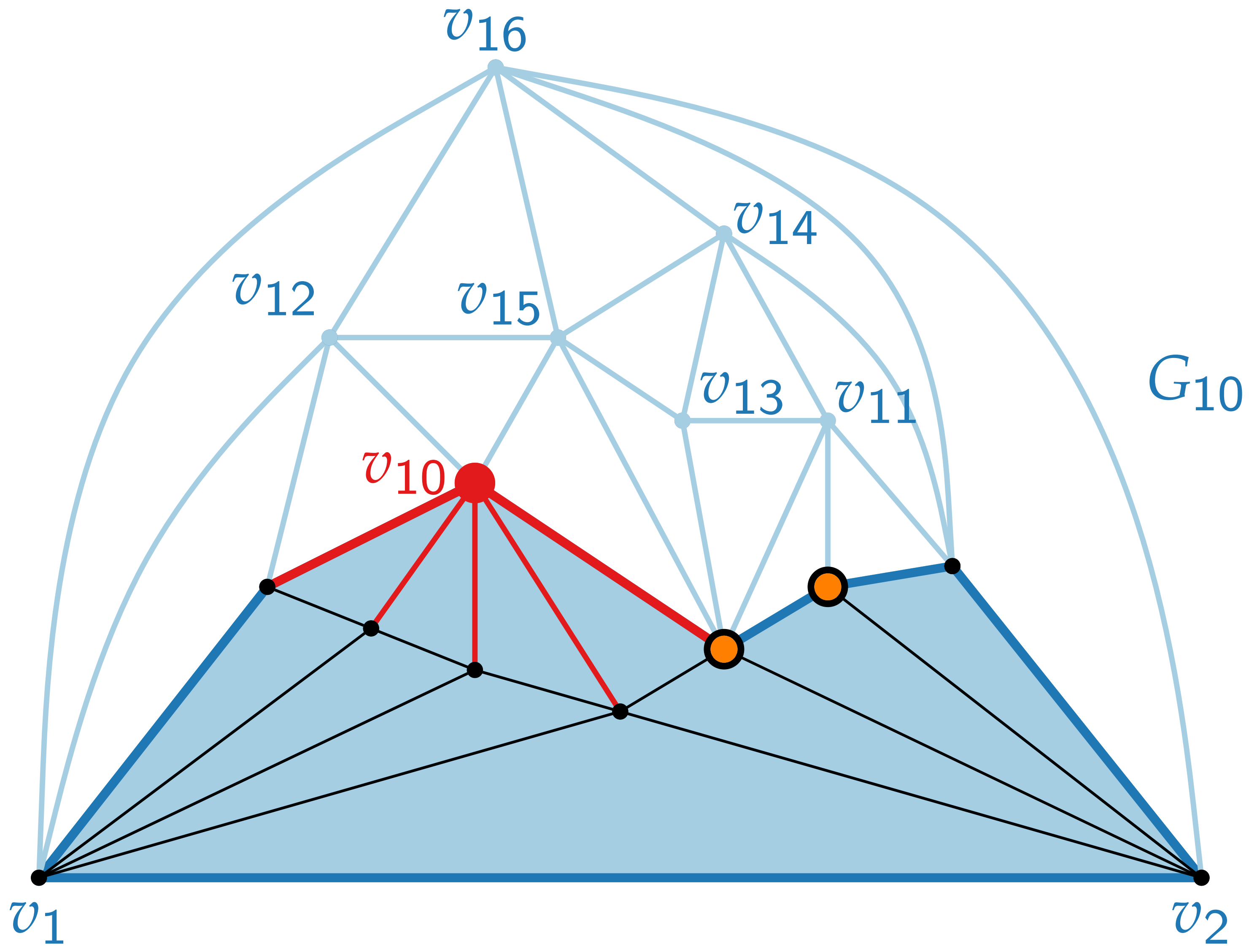

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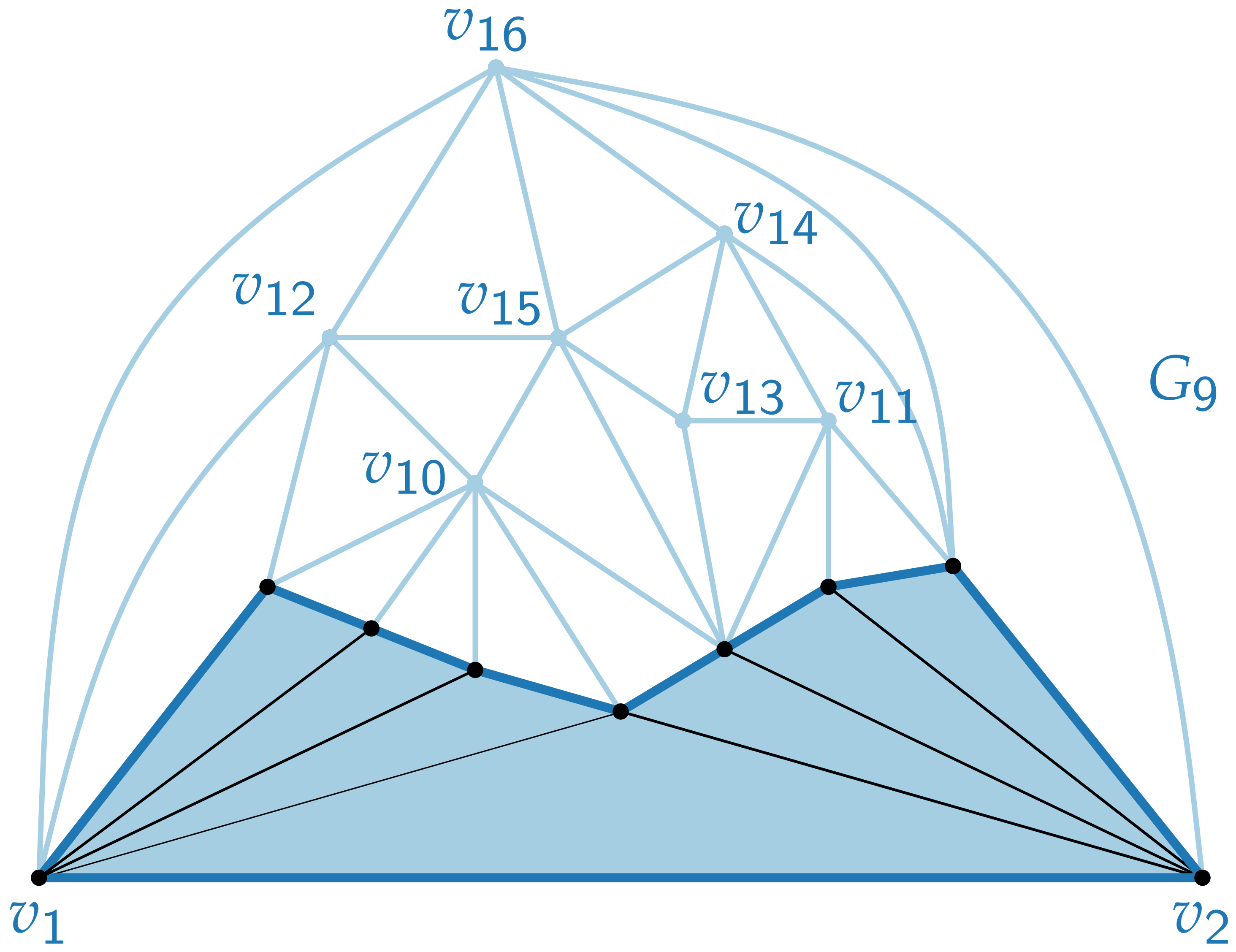

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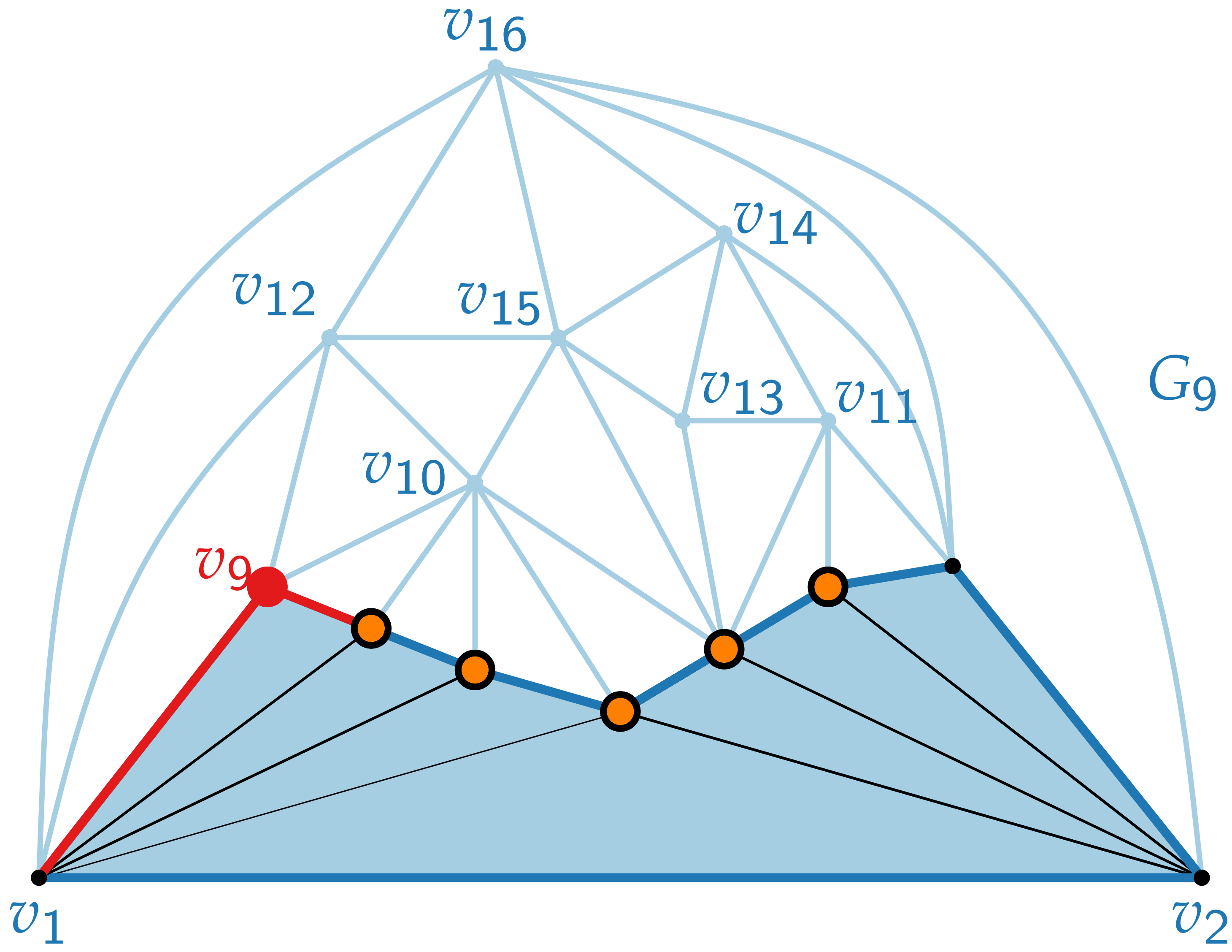

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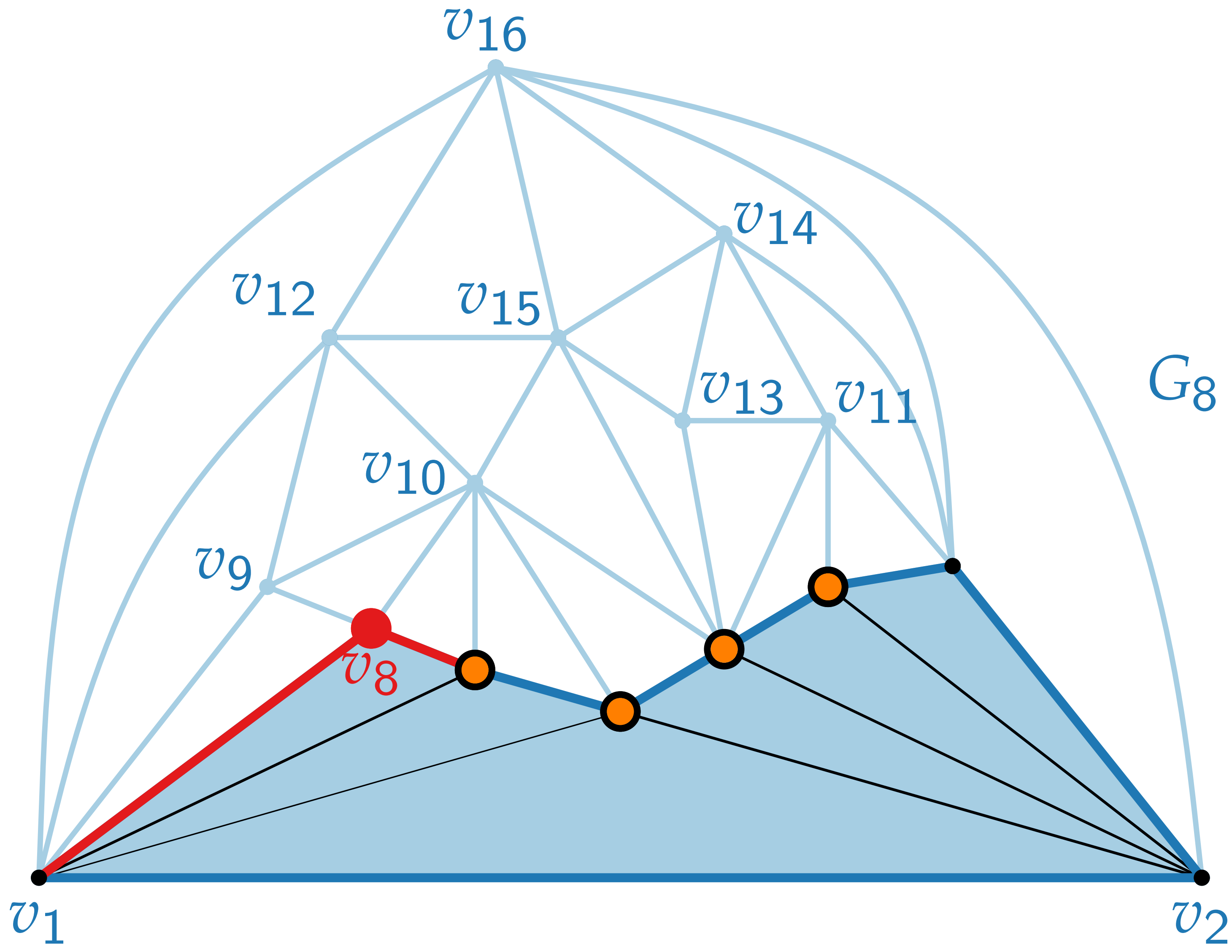

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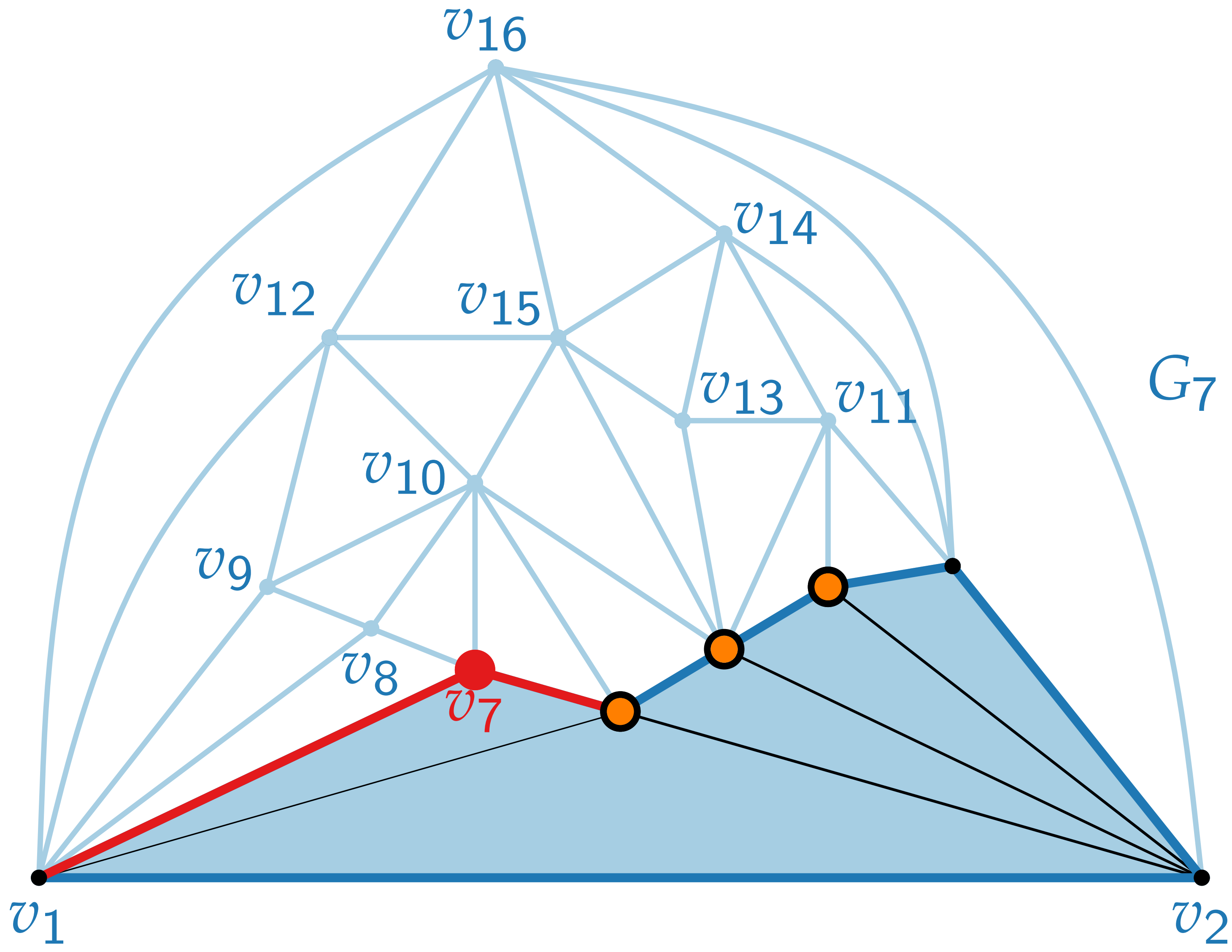

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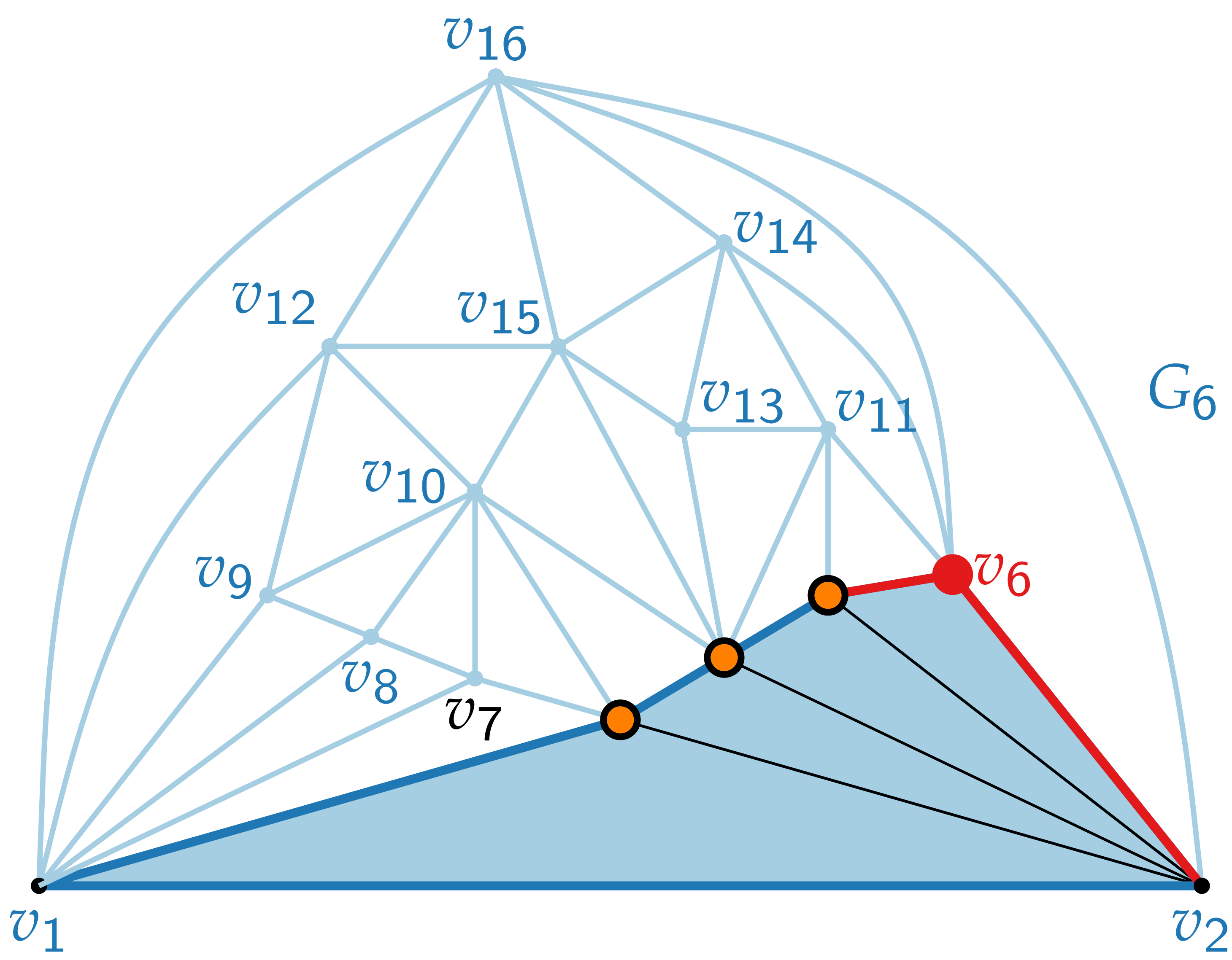

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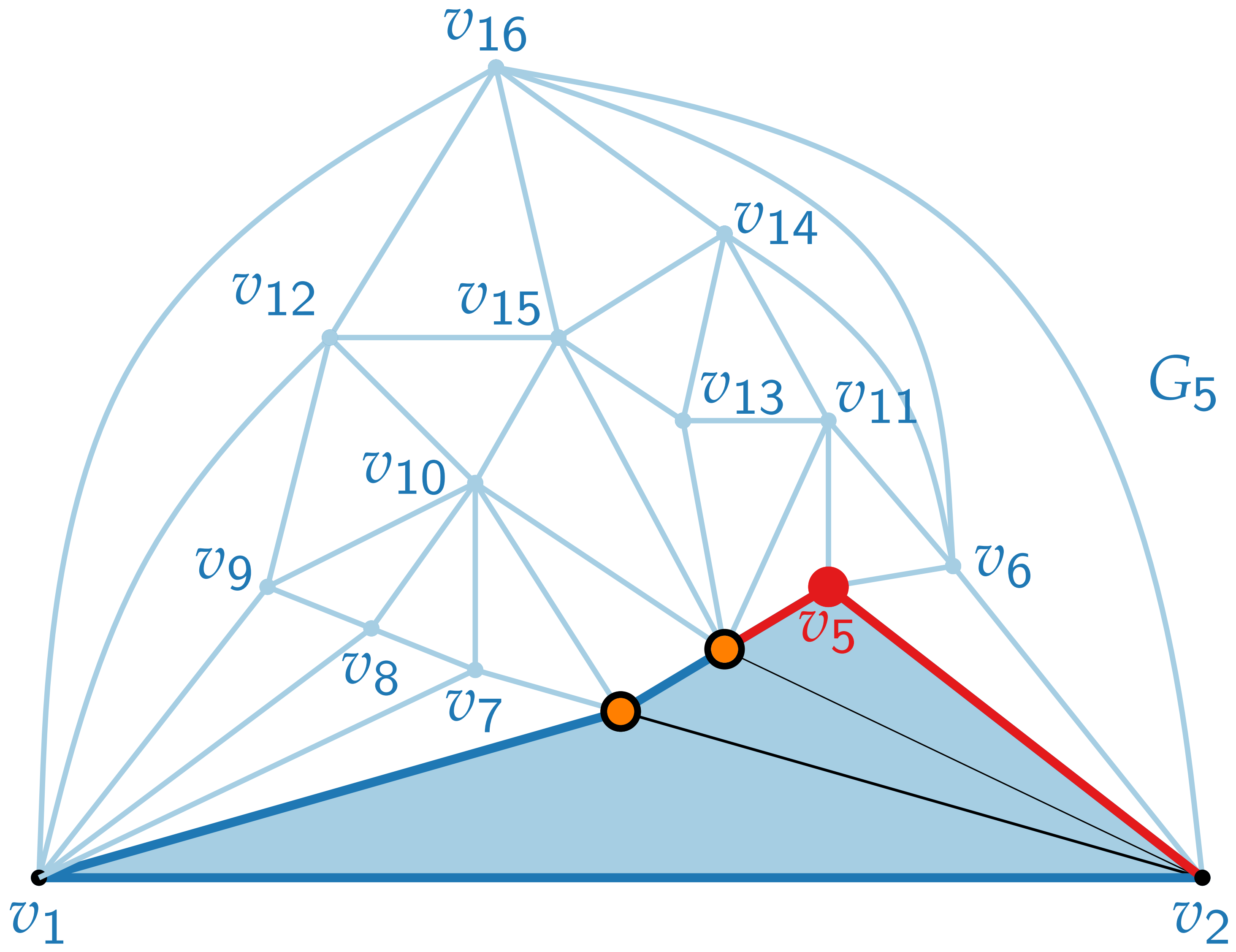

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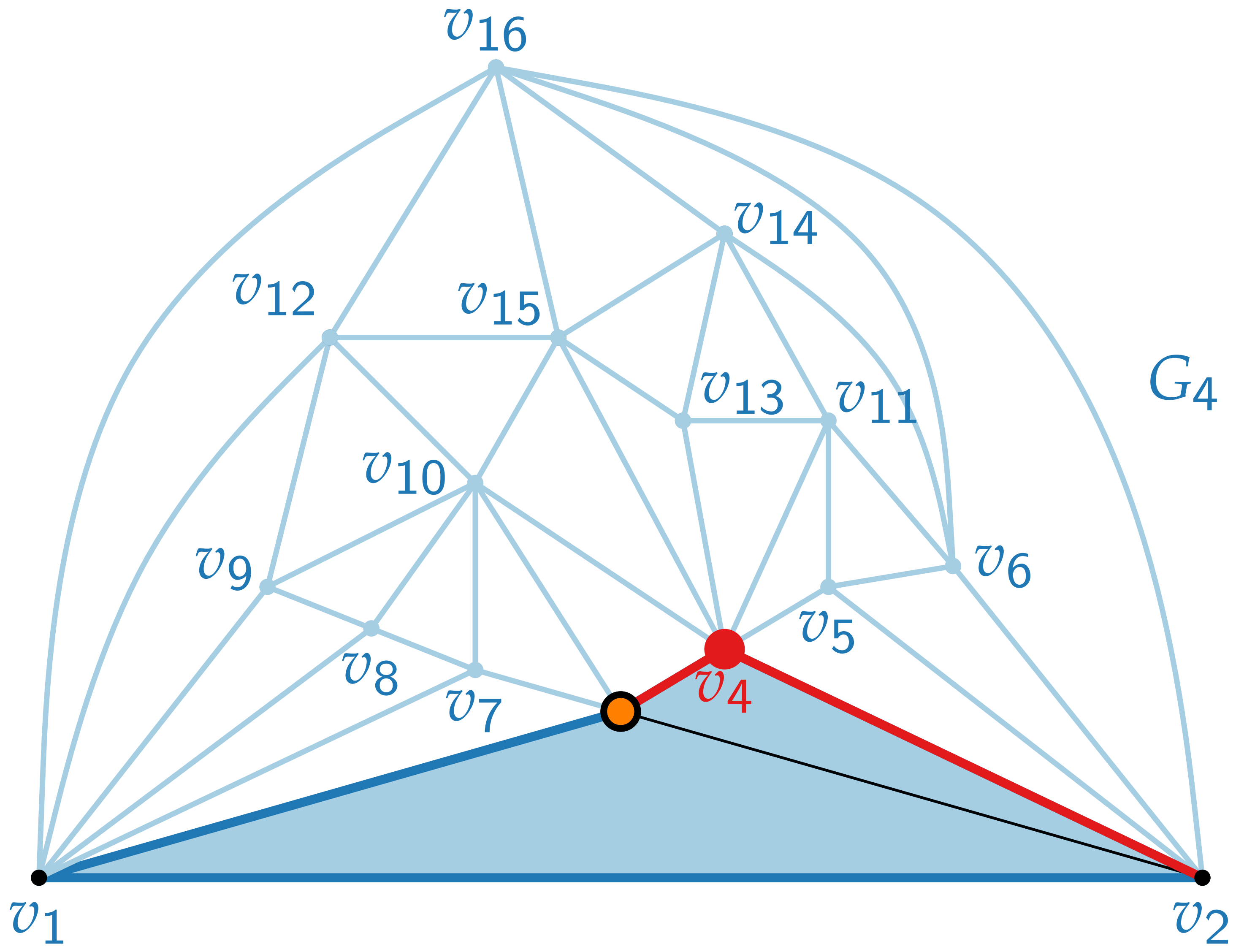

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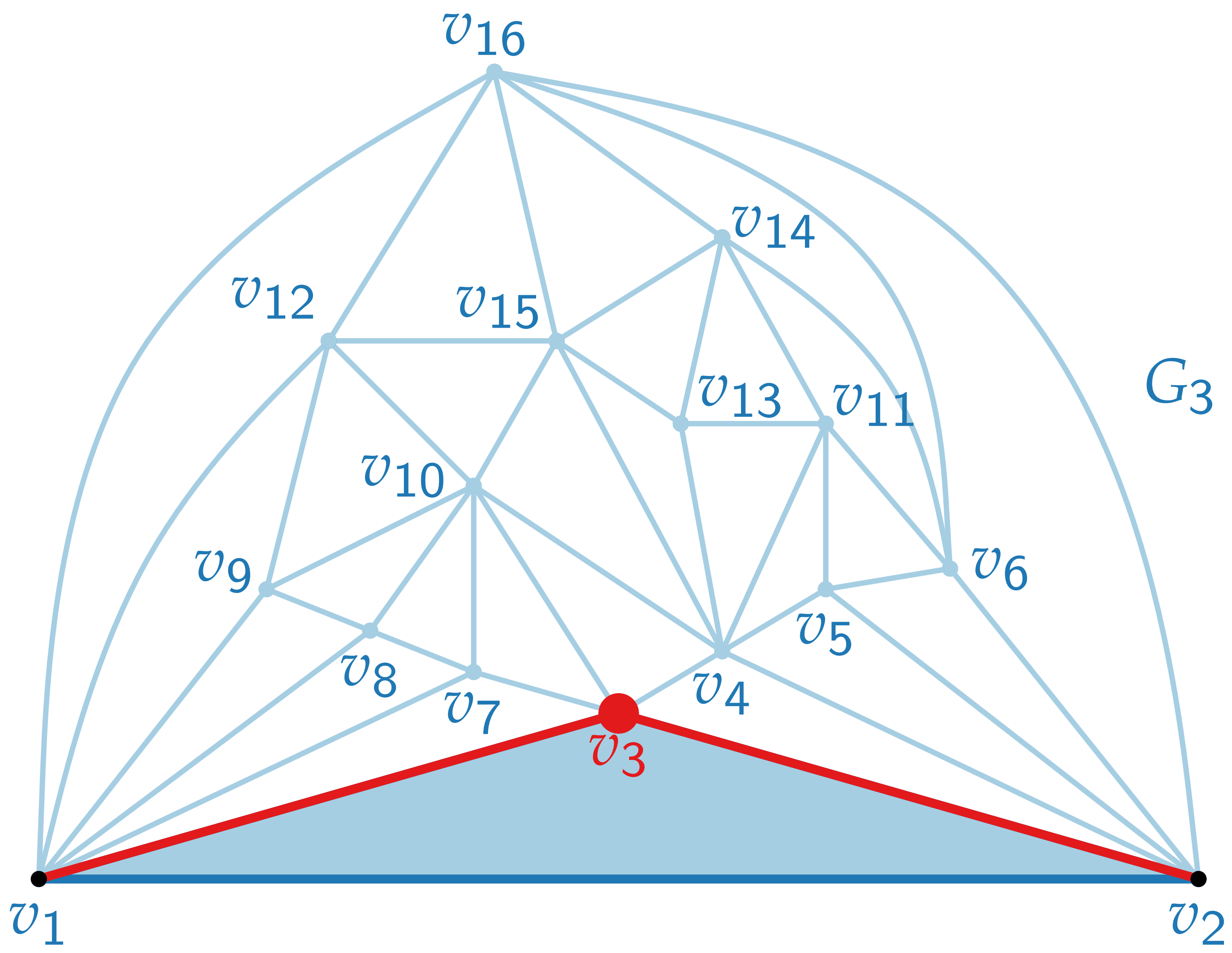

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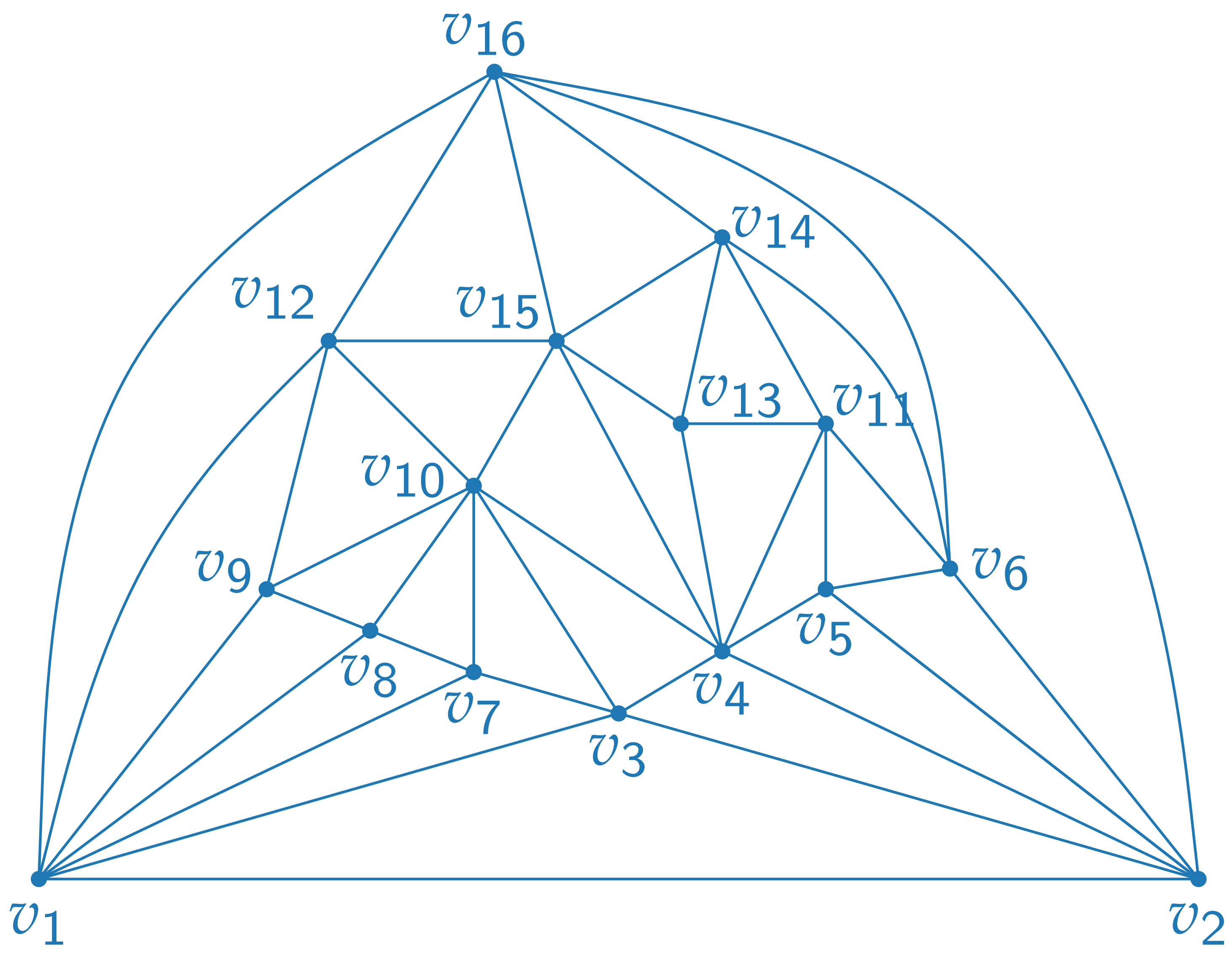

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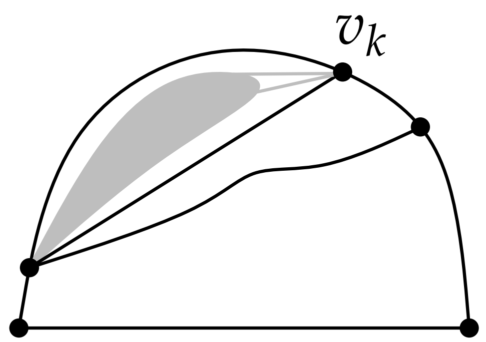

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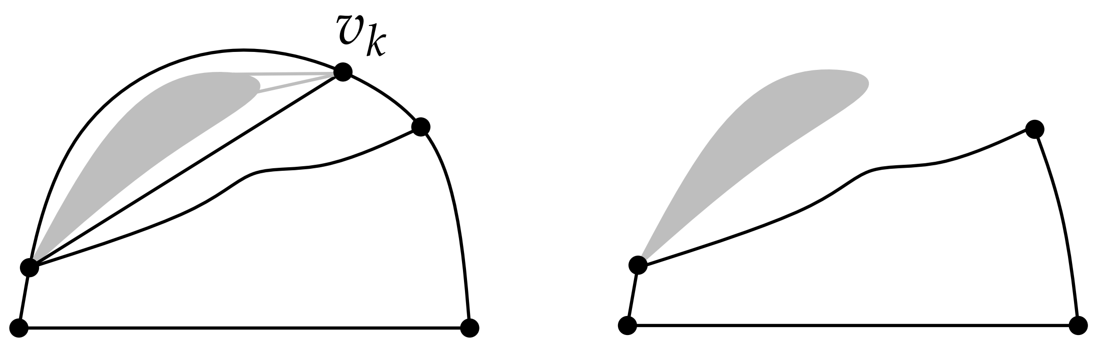

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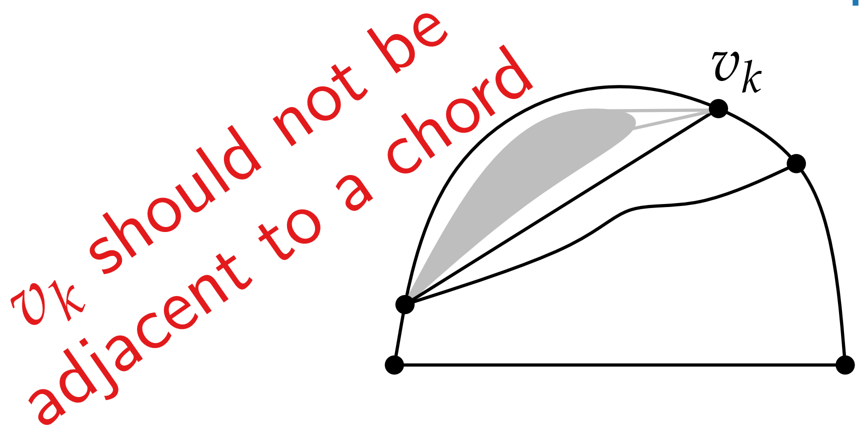

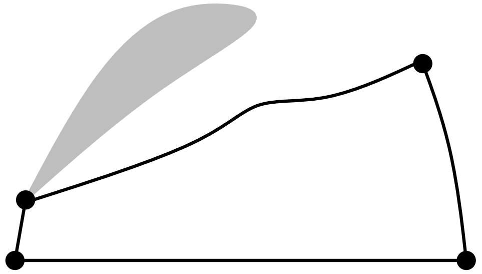

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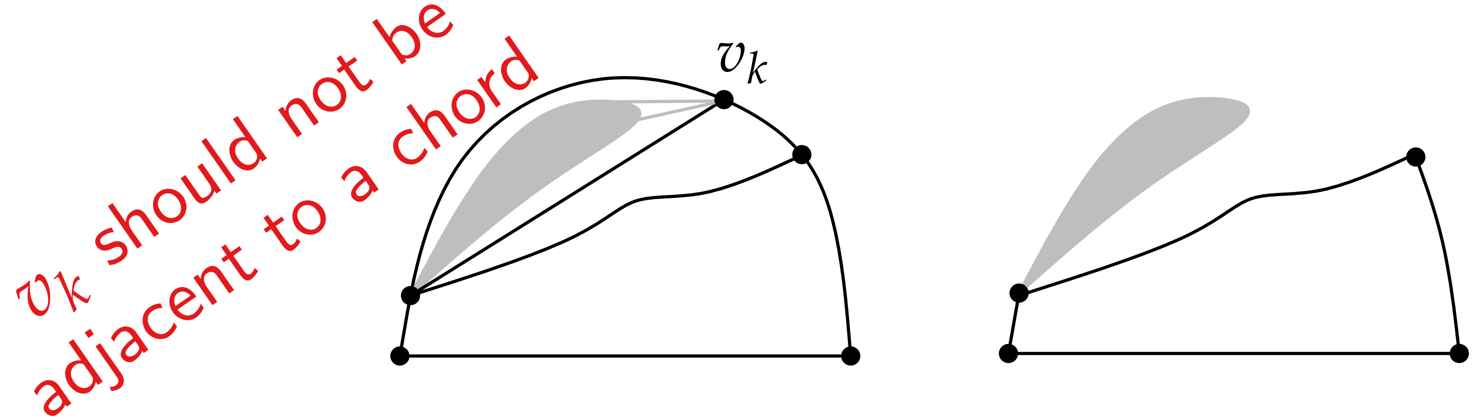

Have to show:

1. v_k not adjacent to chord is sufficient
2. Such v_k exists

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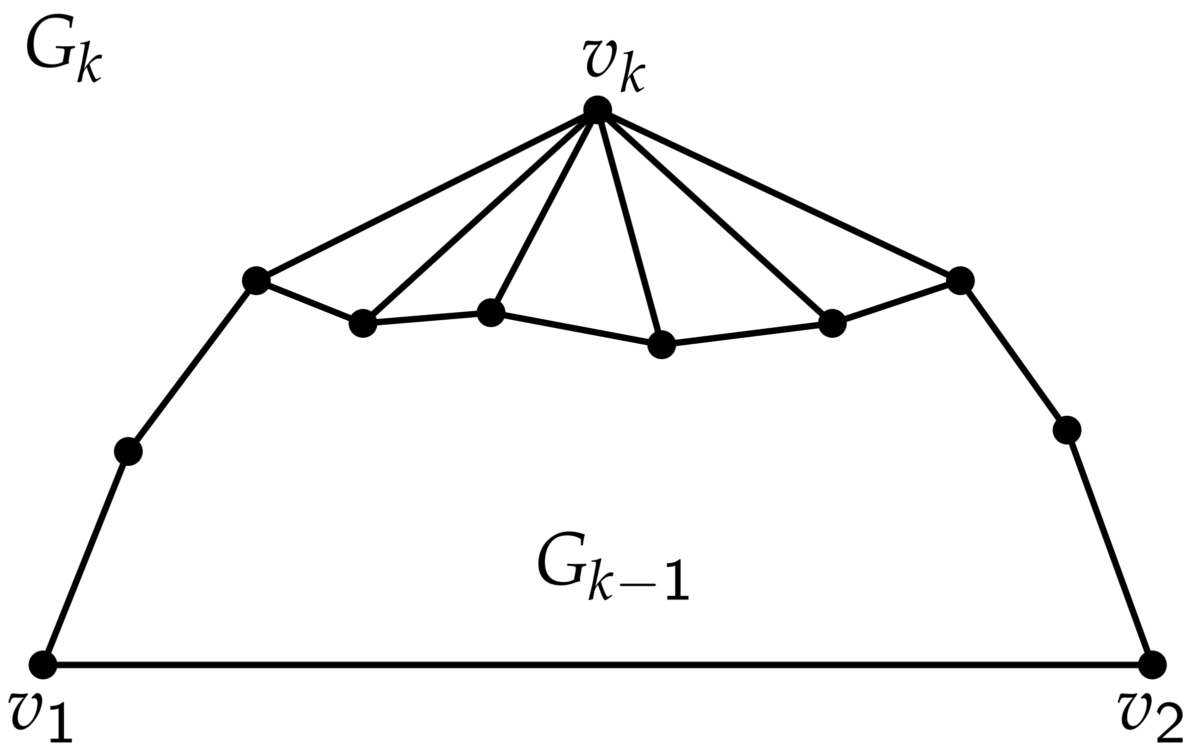

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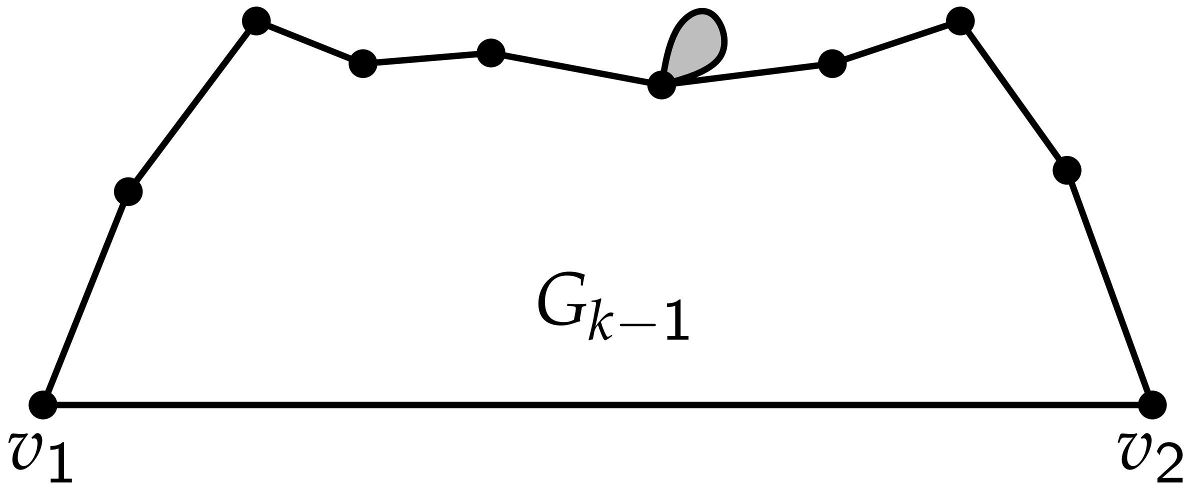

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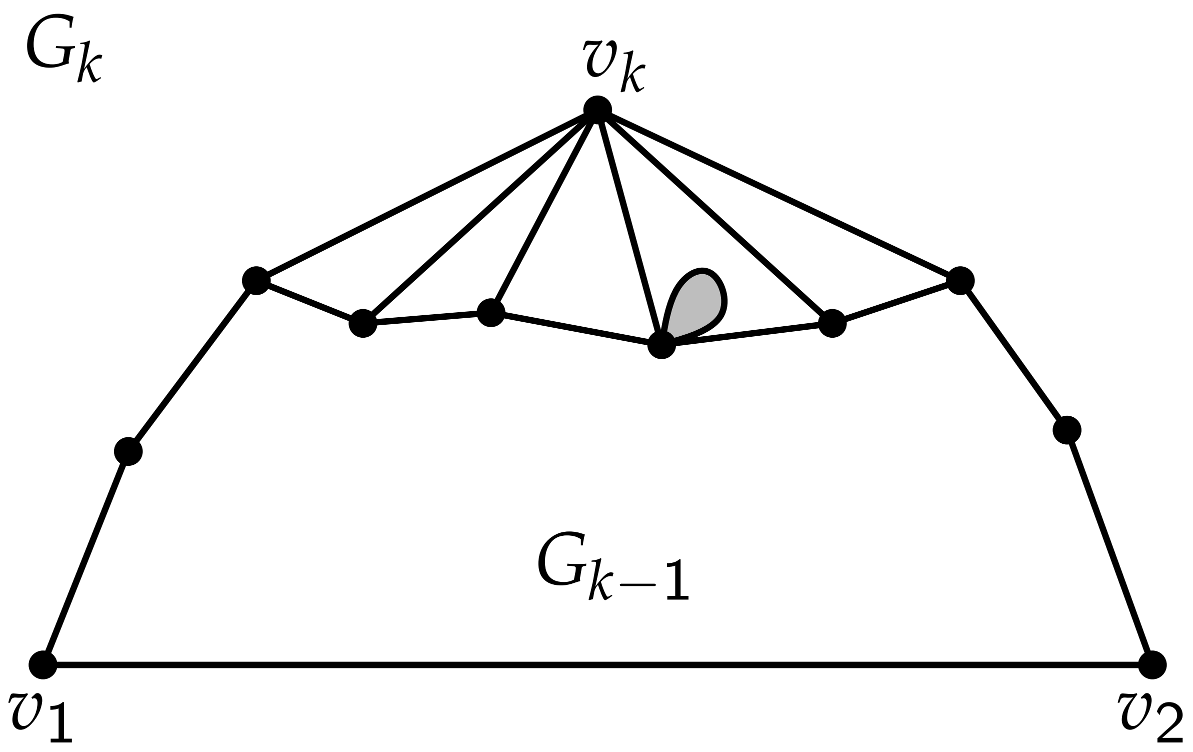

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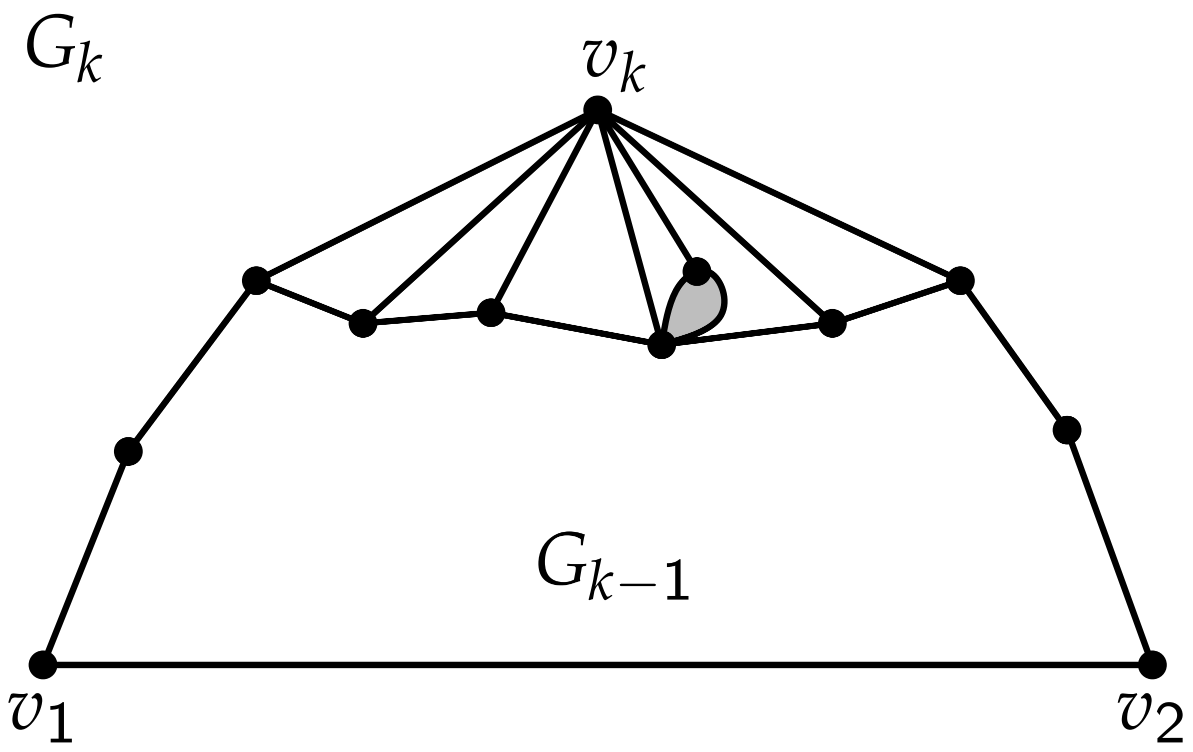

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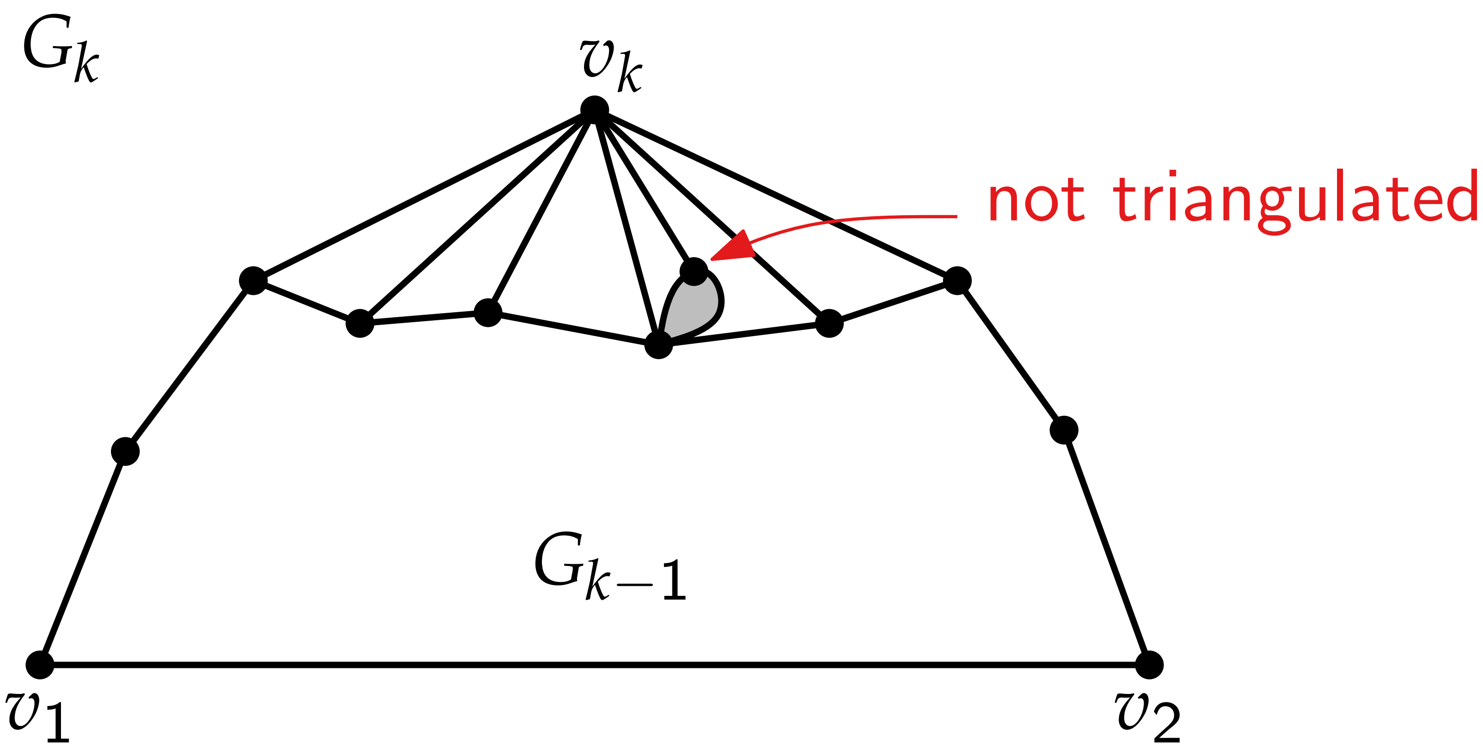

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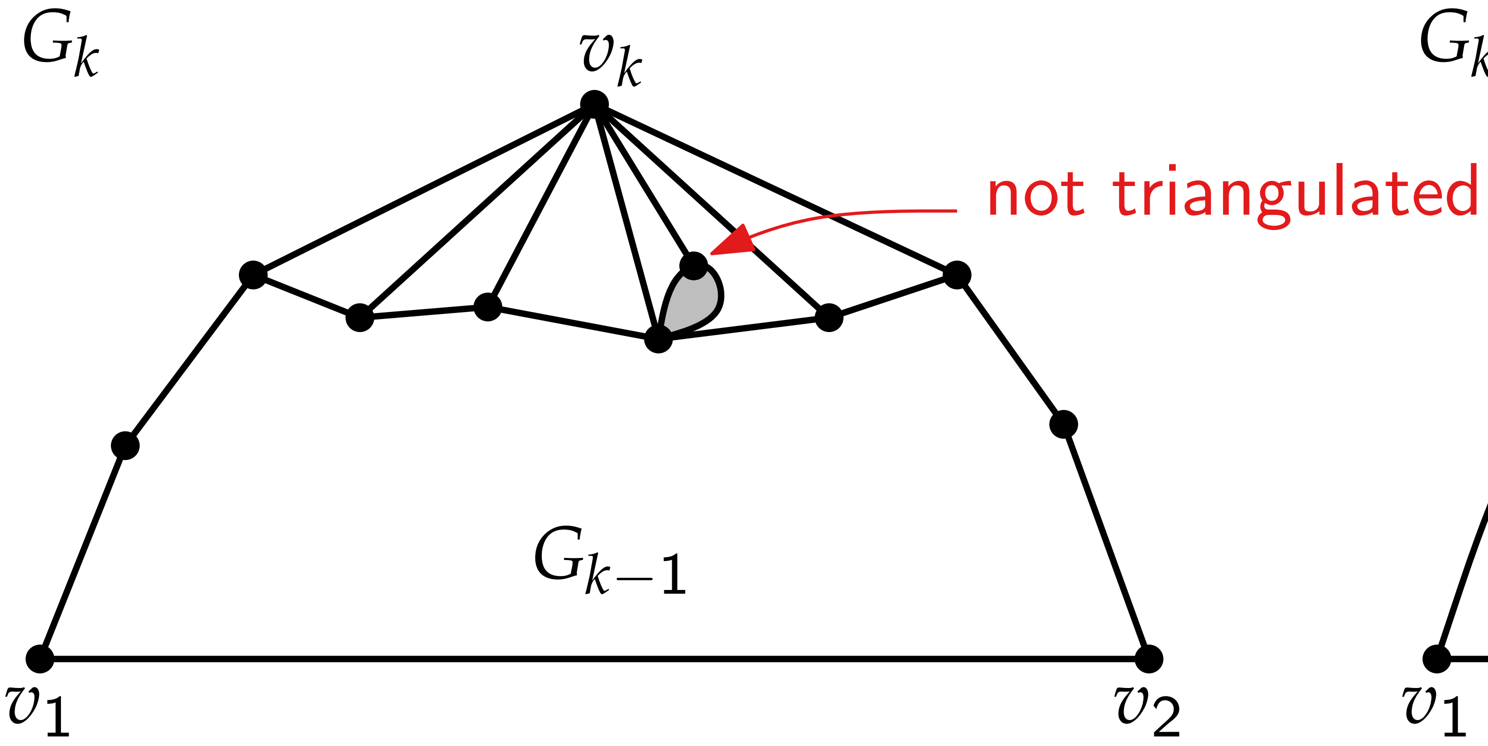

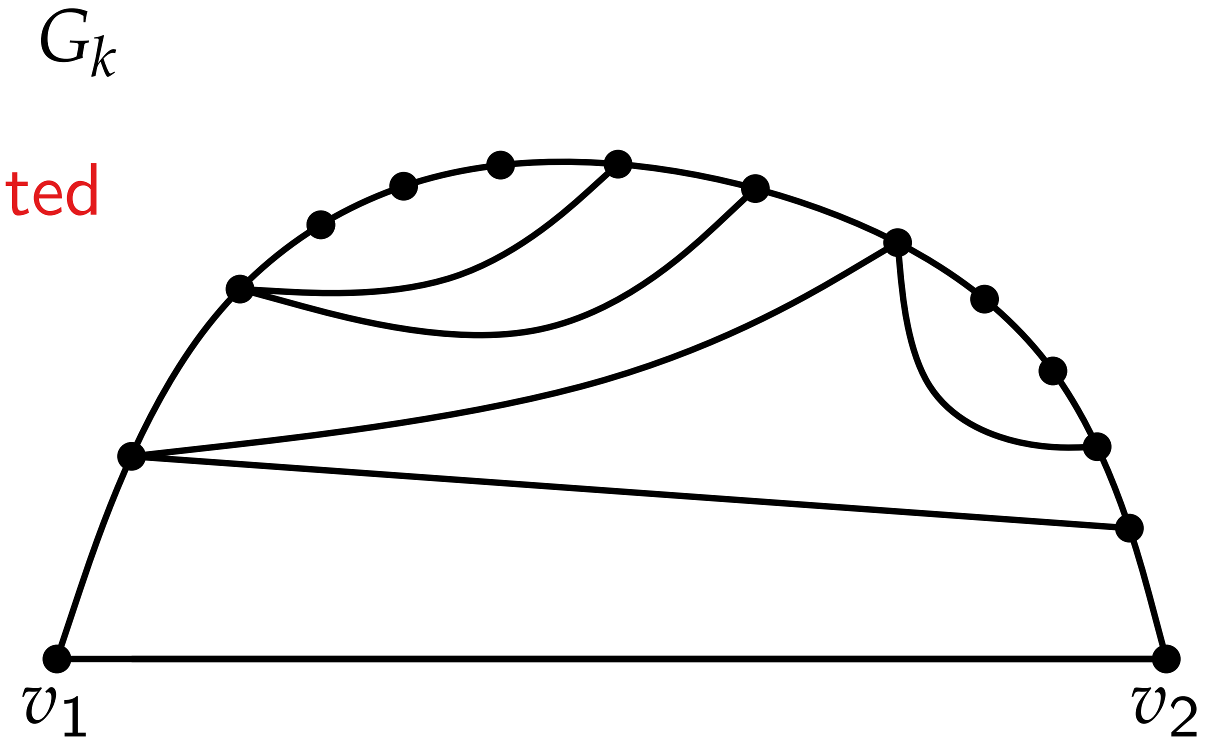

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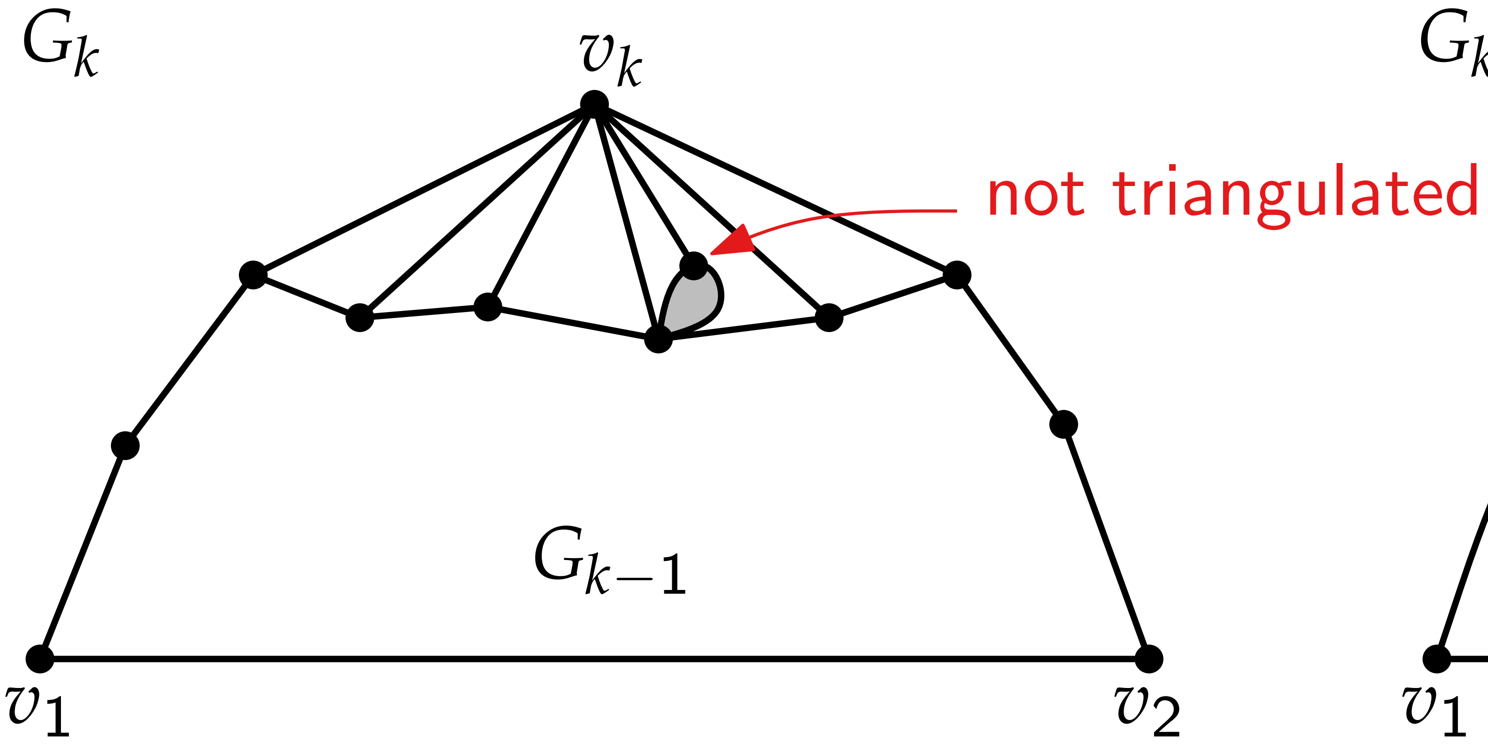

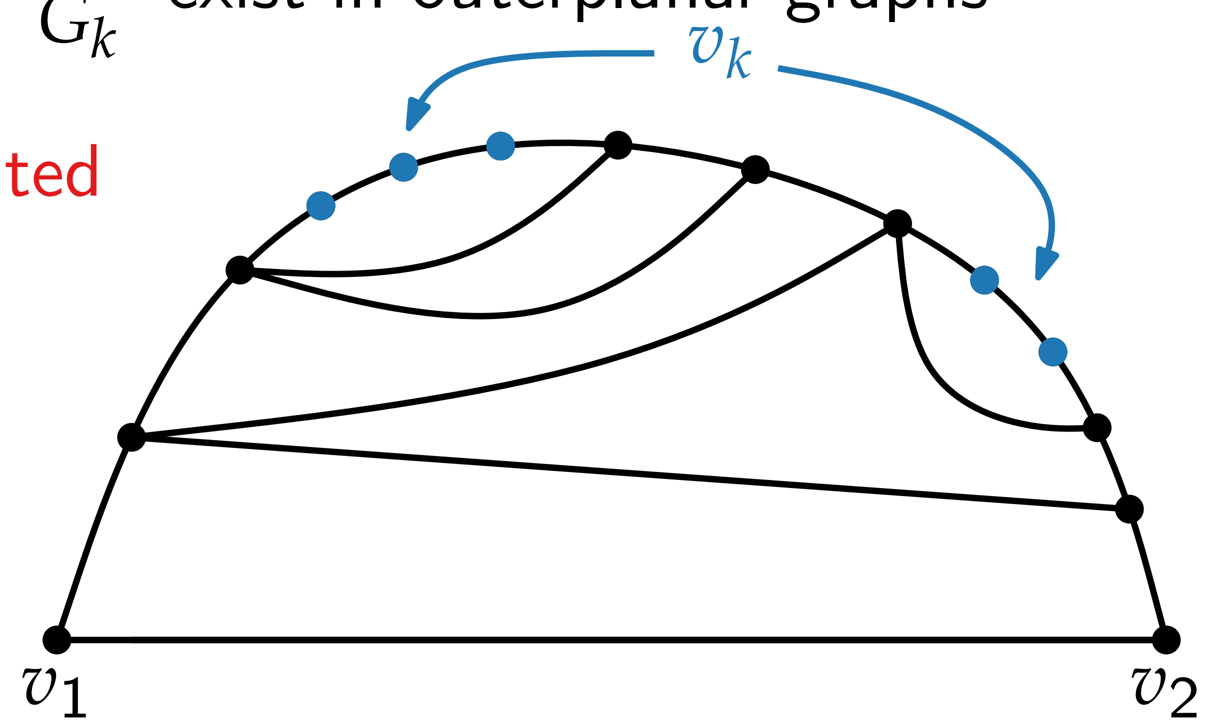

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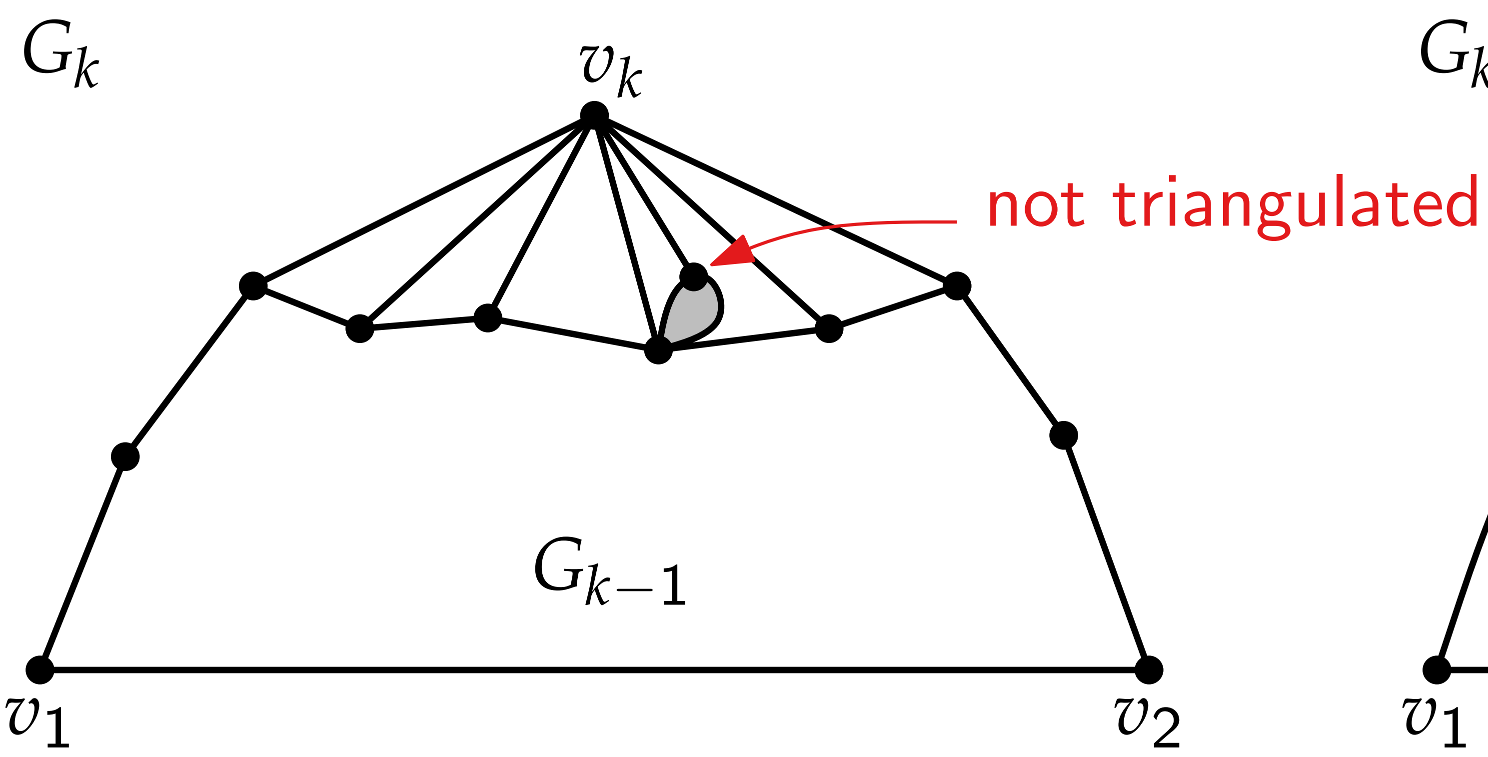
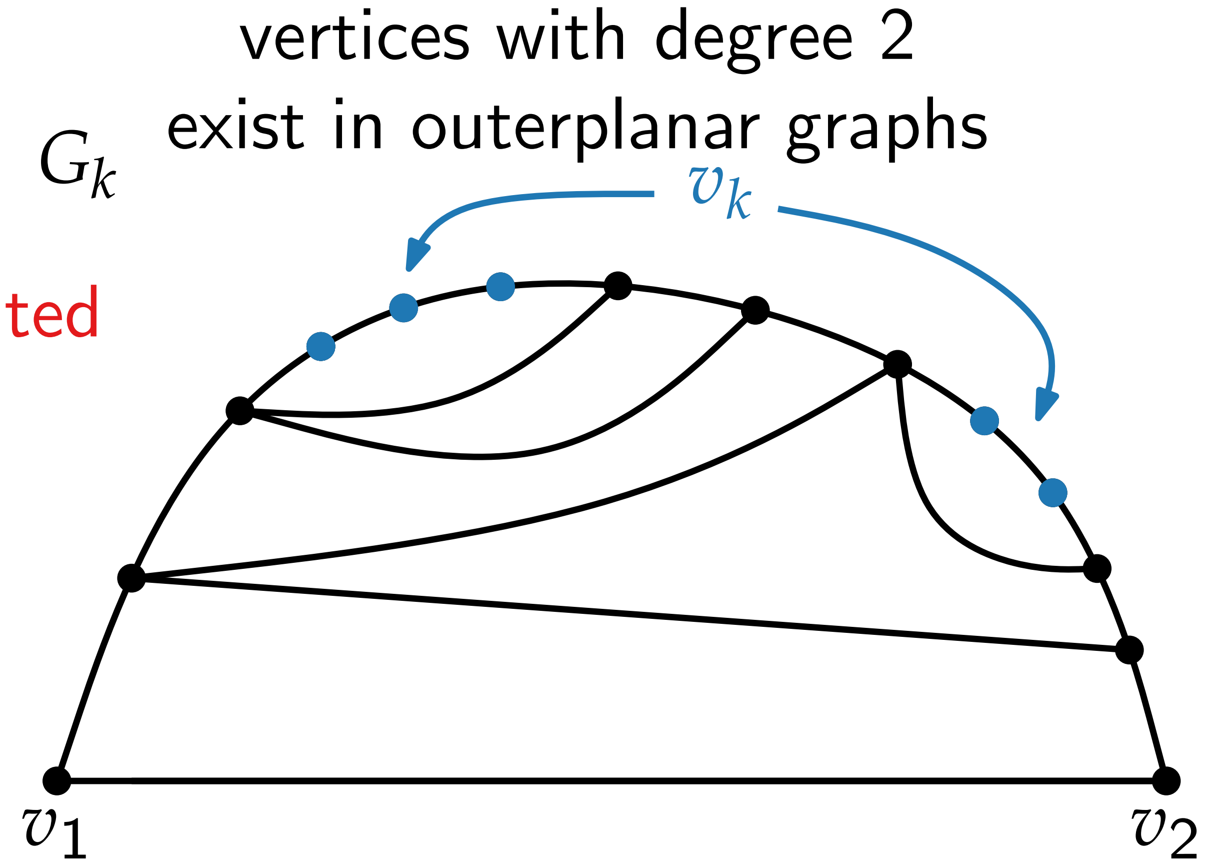
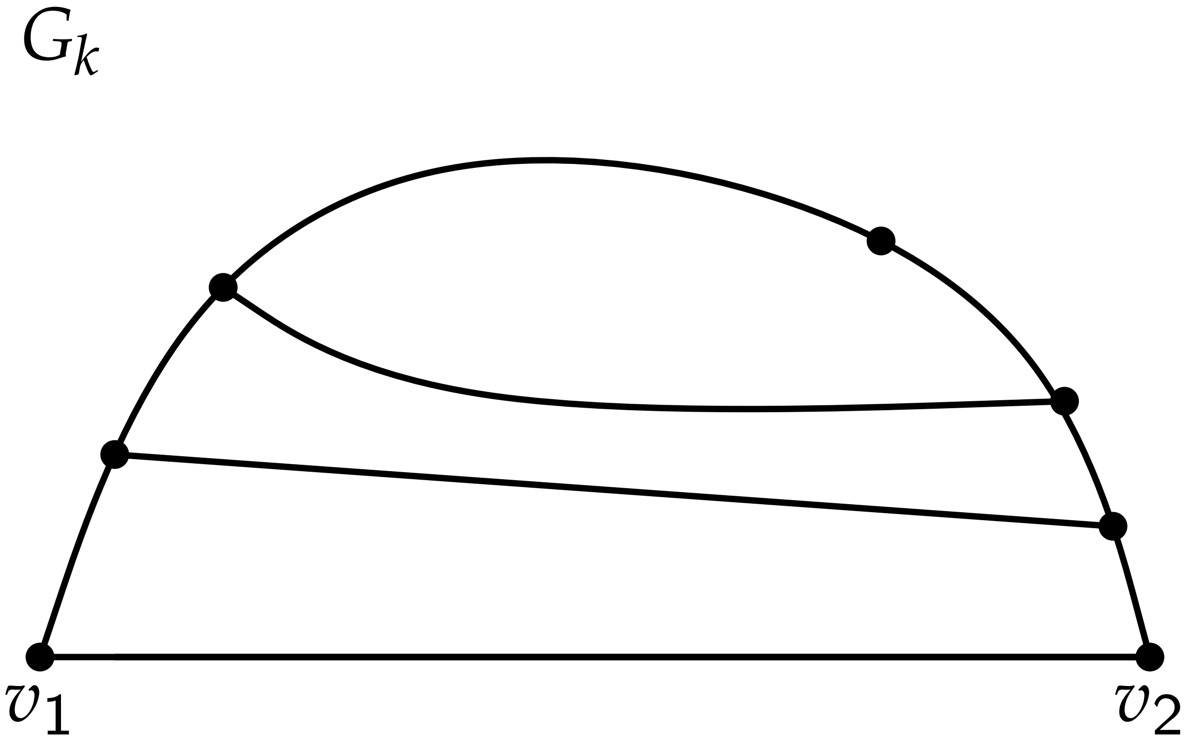




This completes proof of Lemma. □

Canonical order – implementation

- chords of G_k belong to faces:



Canonical order – implementation

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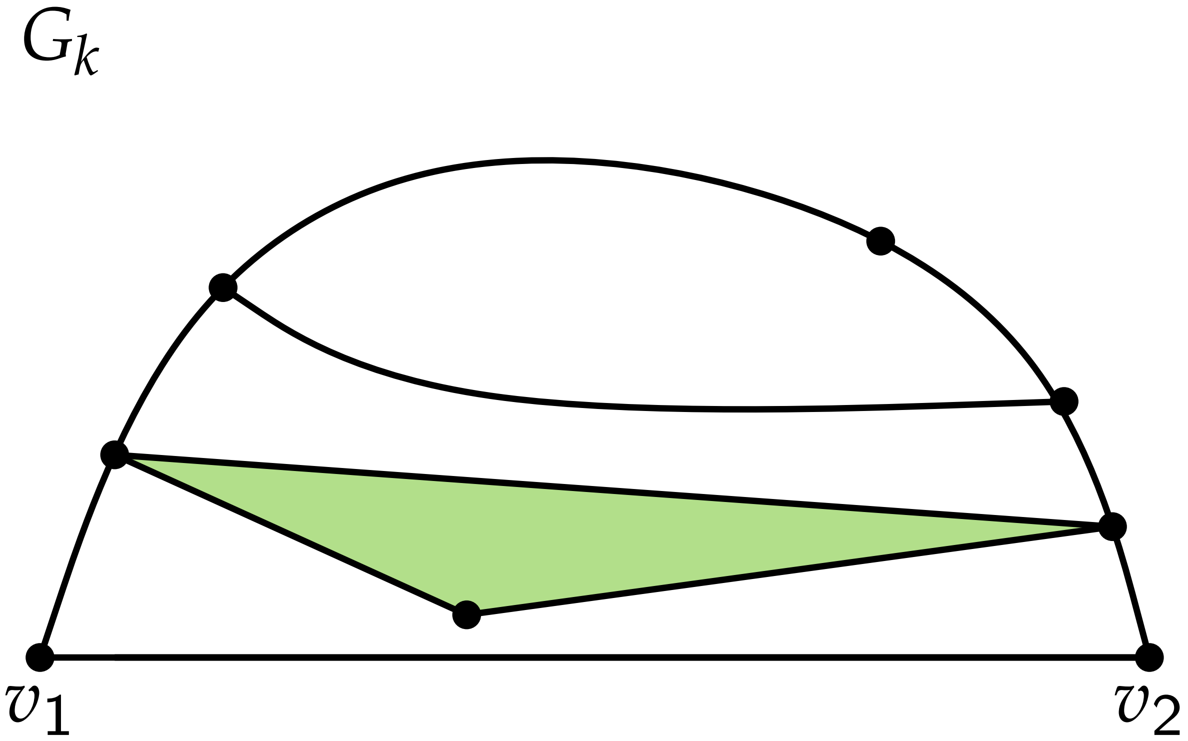

- f has two vertices on the outerface and one internal

Canonical order – implementation

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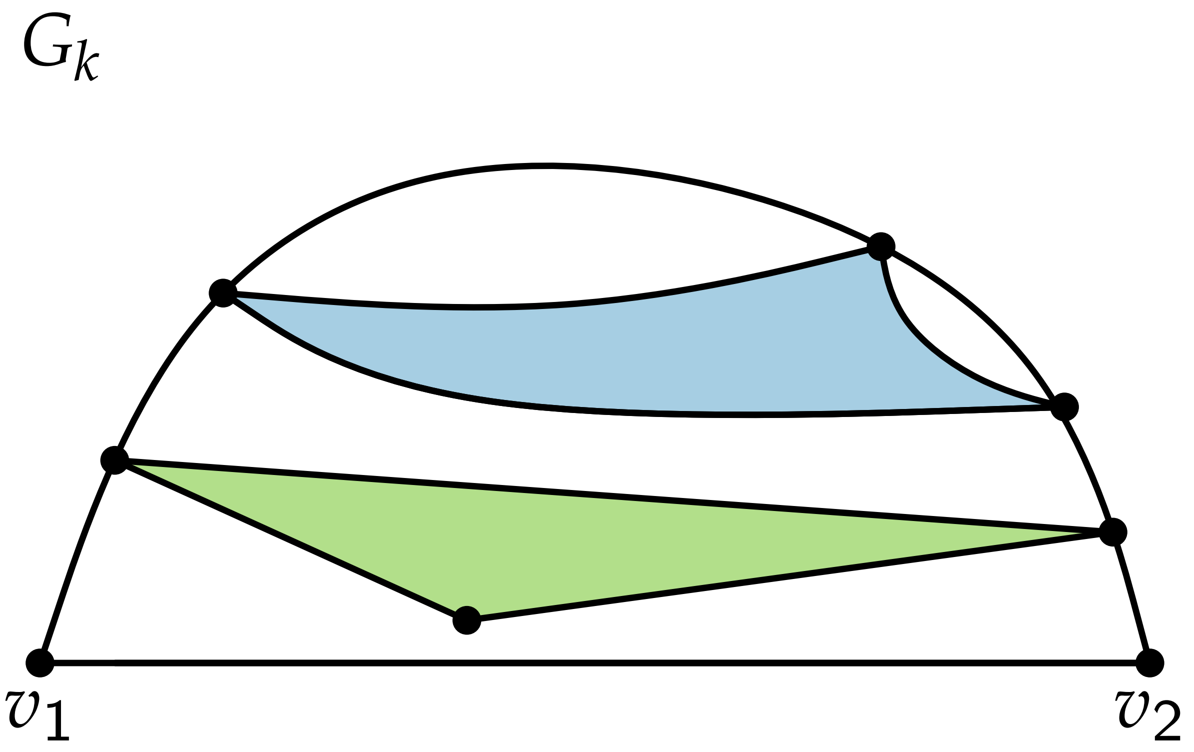

- f has two vertices on the outerface and one internal
- f has three vertices on the outerface and at least two chords

Canonical order – implementation

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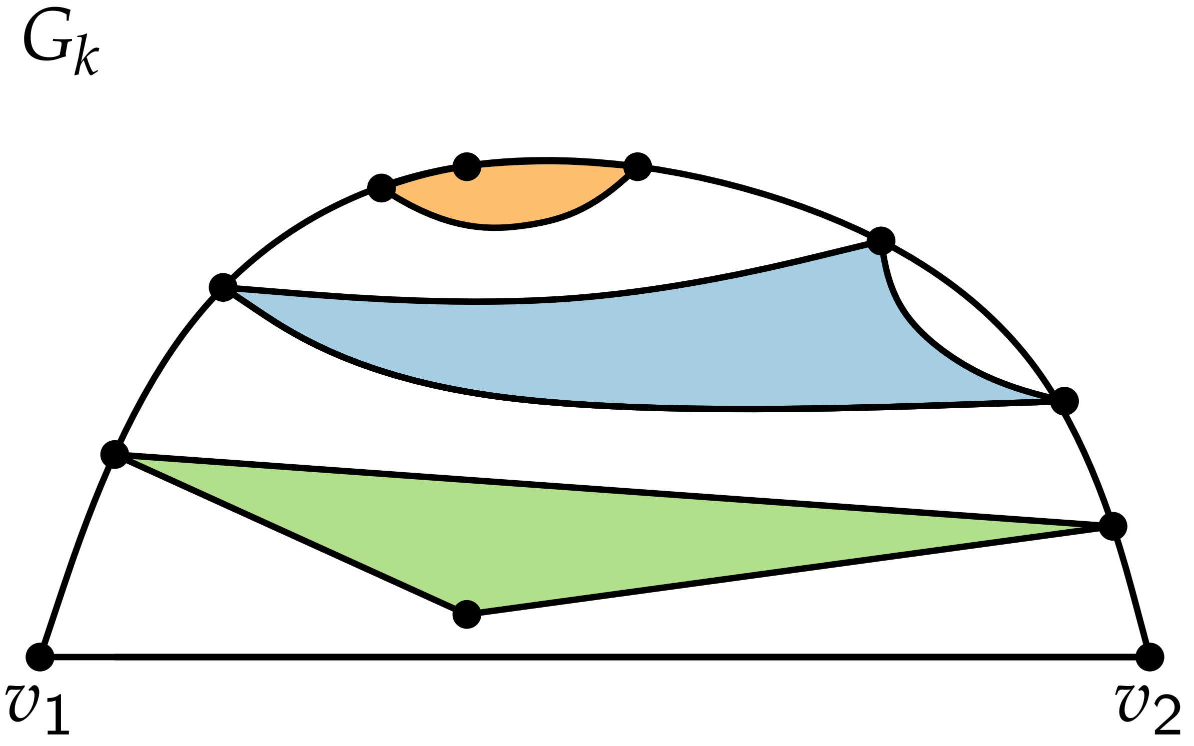

- f has two vertices on the outerface and one internal
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Canonical order – implementation

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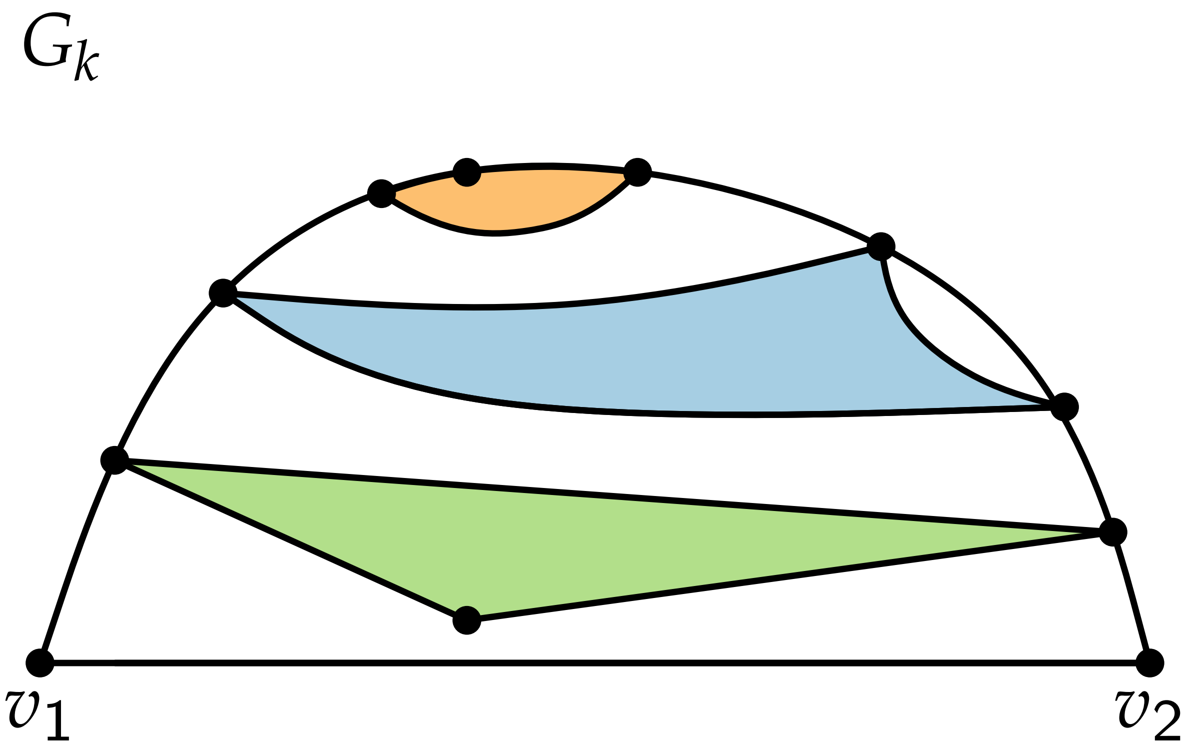

- chords are associated with separating faces
- v_k belongs to no separating faces *

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Canonical order – implementation

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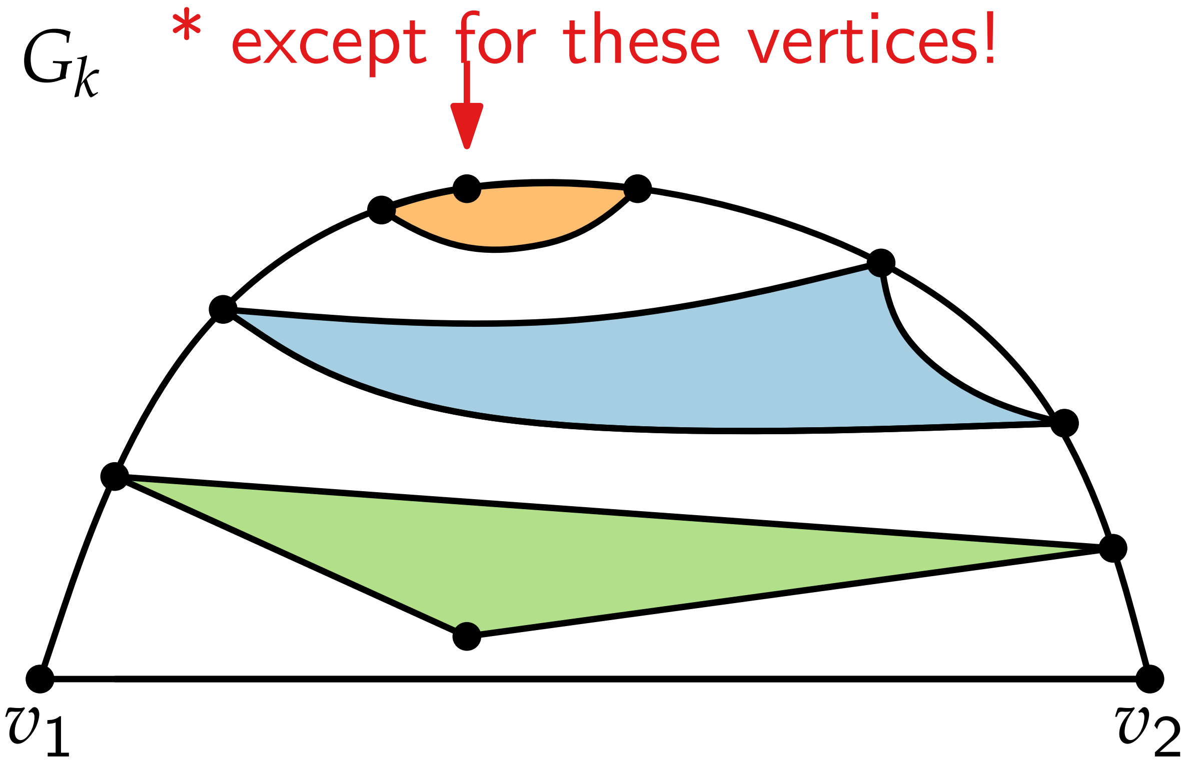

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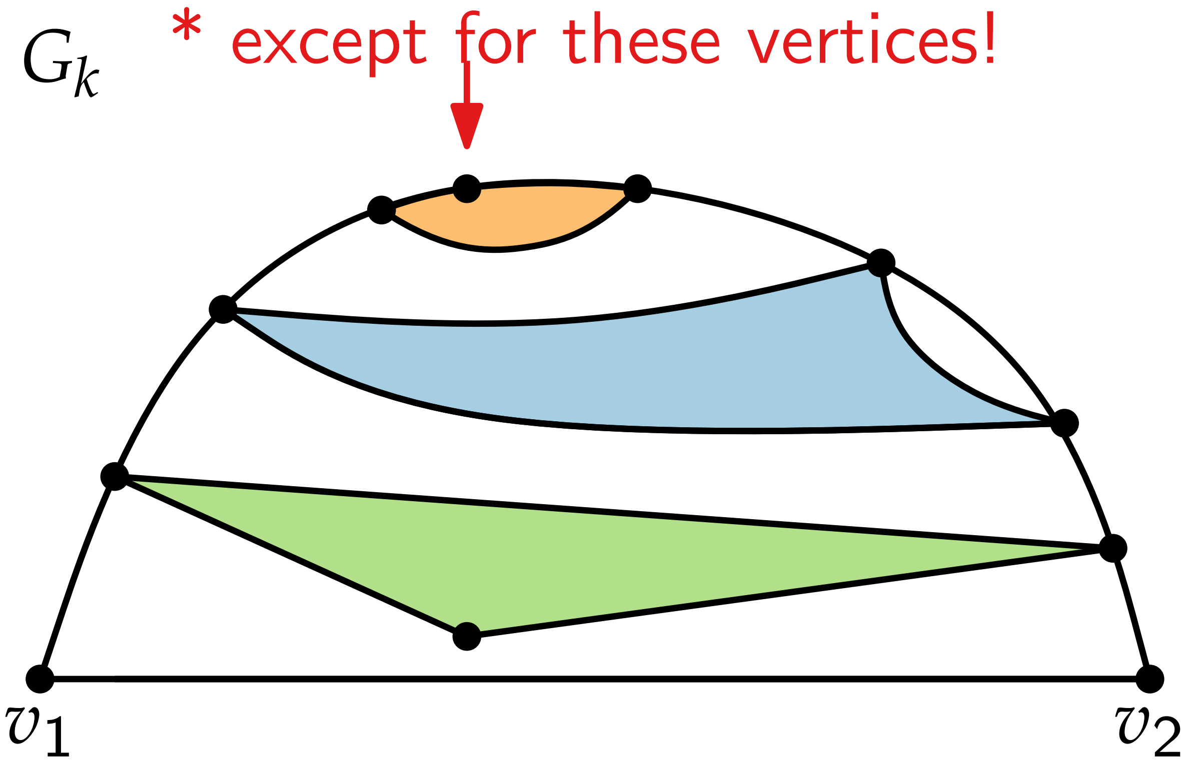
Canonical order – implementation



- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e

- chords are associated with separating faces
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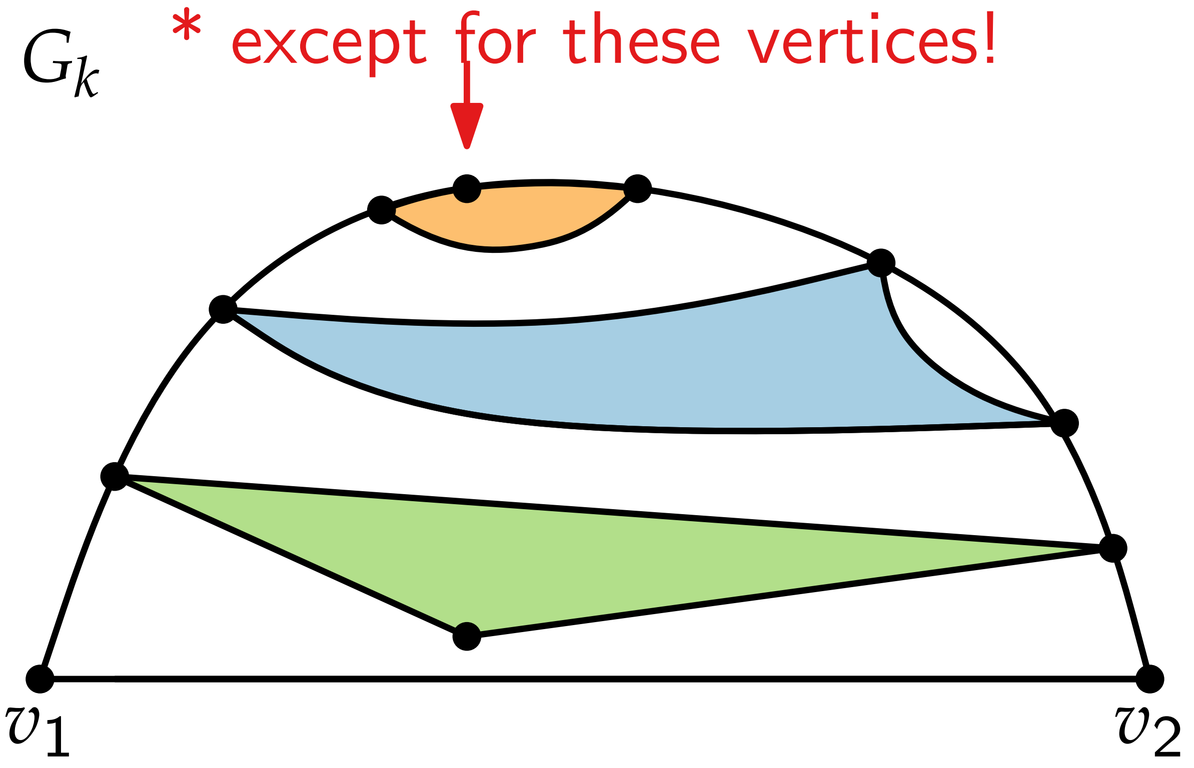
Canonical order – implementation



- chords are associated with separating faces
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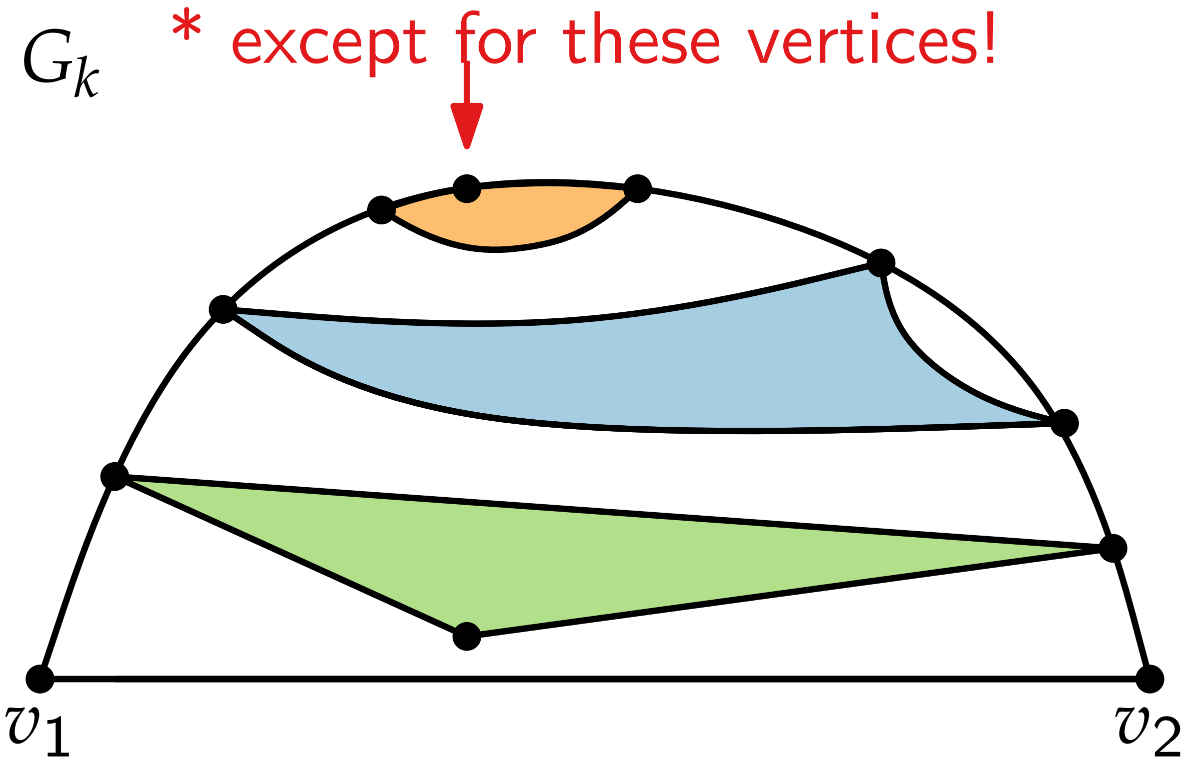
- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
- outE(f) = # edges of f on f_{out}
- sepF(v) = # separation faces that contain v

Canonical order – implementation



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$f \in F(v)$ is separating iff

- outV(f)=3 or
- outV(f)=2 and outE(f)=0

Canonical order – implementation

- f_{out} = current outerface
- $F(v)$ = faces that contain v
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Canonical order – implementation

Algorithm CanonicalOrder- Initialization

forall $v \in V$ **do**
  sepF(v) $\leftarrow$ 0;

forall $f \in F$ **do**
  outV(f), outE(f) $\leftarrow$ 0;

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
- outE(f) = # edges of f on f_{out}
- sepF(v) = # separation faces that contain v

Canonical order – implementation

Algorithm CanonicalOrder- Initialization

forall $v \in V$ **do**
  sepF(v) $\leftarrow 0$;

forall $f \in F$ **do**
  outV(f), outE(f) $\leftarrow 0$;

forall $v \in f_{out}$ **do**
  forall $f \in F(v)$: $f \neq f_{out}$ **do**
    outV(f)$++$;

forall $e \in f_{out}$ **do**
  forall $f \in F(e)$: $f \neq f_{out}$ **do**
    outE(f)$++$;

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
- outE(f) = # edges of f on f_{out}
- sepF(v) = # separation faces that contain v

Canonical order – implementation

Algorithm CanonicalOrder- Initialization

forall $v \in V$ **do**
  sepF(v) $\leftarrow$ 0;

forall $f \in F$ **do**
  outV(f), outE(f) $\leftarrow$ 0;

forall $v \in f_{out}$ **do**
  forall $f \in F(v)$: $f \neq f_{out}$ **do**
    outV(f)++;

forall $e \in f_{out}$ **do**
  forall $f \in F(e)$: $f \neq f_{out}$ **do**
    outE(f)++;

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
- outE(f) = # edges of f on f_{out}
- sepF(v) = # separation faces that contain v

forall $v \in f_{out}$ **do**
  forall $f \in F(v)$: $f \neq f_{out}$ **do**
    if outV(f)=3 or outV(f)=2 and outE(f)=0 **then**
      sepF(v)++;

Canonical order – implementation

Remove degree 2 vertex v_k

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
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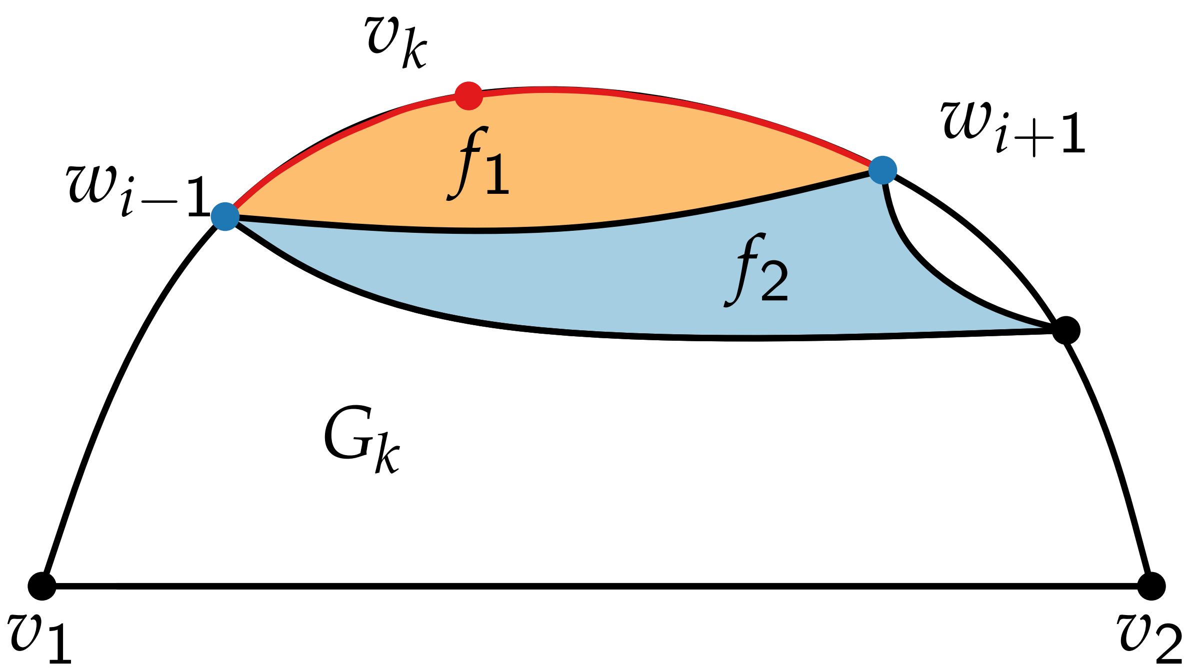
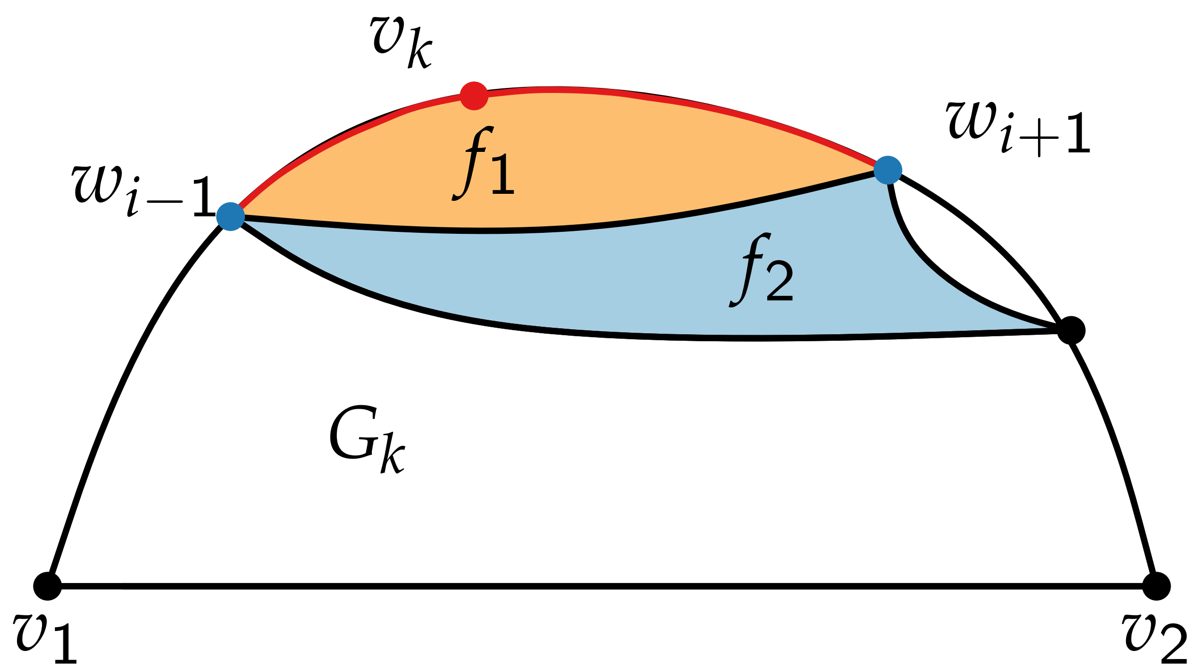


Canonical order – implementation

Remove degree 2 vertex v_k

- v_k and f_1 are removed
- outE(f_2) increases by one
- sepF(w_{i-1}) decreases by one
- sepF(w_{i+1}) decreases by one

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
- outE(f) = # edges of f on f_{out}
- sepF(v) = # separation faces that contain v



Canonical order – implementation

Remove degree 2 vertex v_k

- v_k and f_1 are removed
- outE(f_2) increases by one
- sepF(w_{i-1}) decreases by one
- sepF(w_{i+1}) decreases by one

- if f_2 has outV(f_2)=2, f_2 is not a separating face
  - sepF(w_{i-1}) decreases by one
  - sepF(w_{i+1}) decreases by one

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
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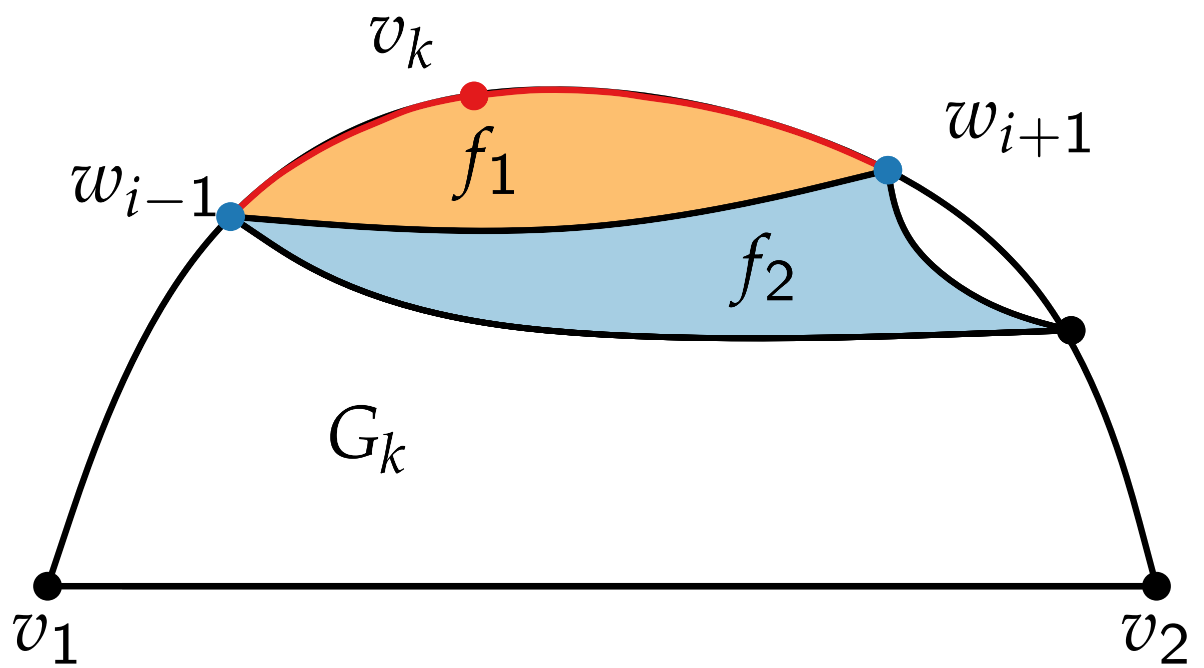

Canonical order – implementation

Remove v_k with sepF(v_k)$= 0$

- f_{out} = current outerface
- $F(v)$ = faces that contain v
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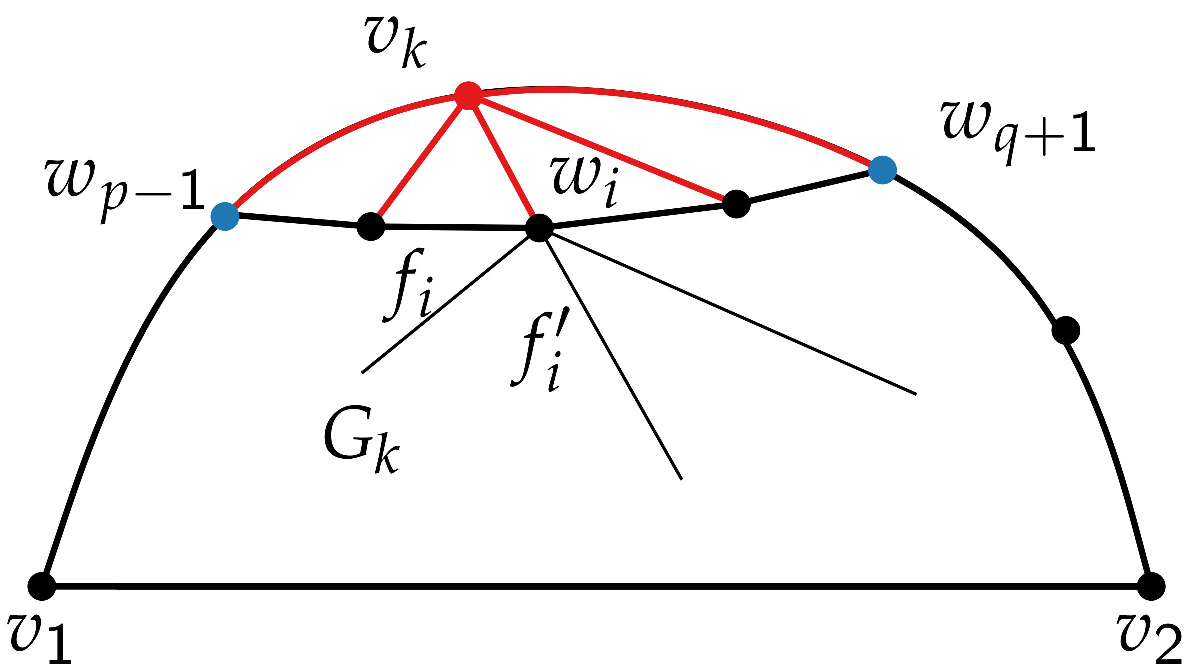
- outV(f) = # vertices of f on f_{out}
- outE(f) = # edges of f on f_{out}
- sepF(v) = # separation faces that contain v

- face f_i contains edge (w_{i-1}, w_i) of the outerface of G_{k-1}
- face f_i' contains edges of w_i that are in the interior of G_{k-1}



Canonical order – implementation

Remove v_k with sepF(v_k)$= 0$

- v_k and faces that contain v_k are removed
- outV(f_i) increases by two, $p+1 \leq i \leq q$
- outV(f_p), outV(f_{q+1}) increases by one
- outV(f_i') incrases by one, $p \leq i \leq q$
- outE(f_i) increases by one, $p \leq i \leq q+1$

- f_{out} = current outerface
- $F(v)$ = faces that contain v
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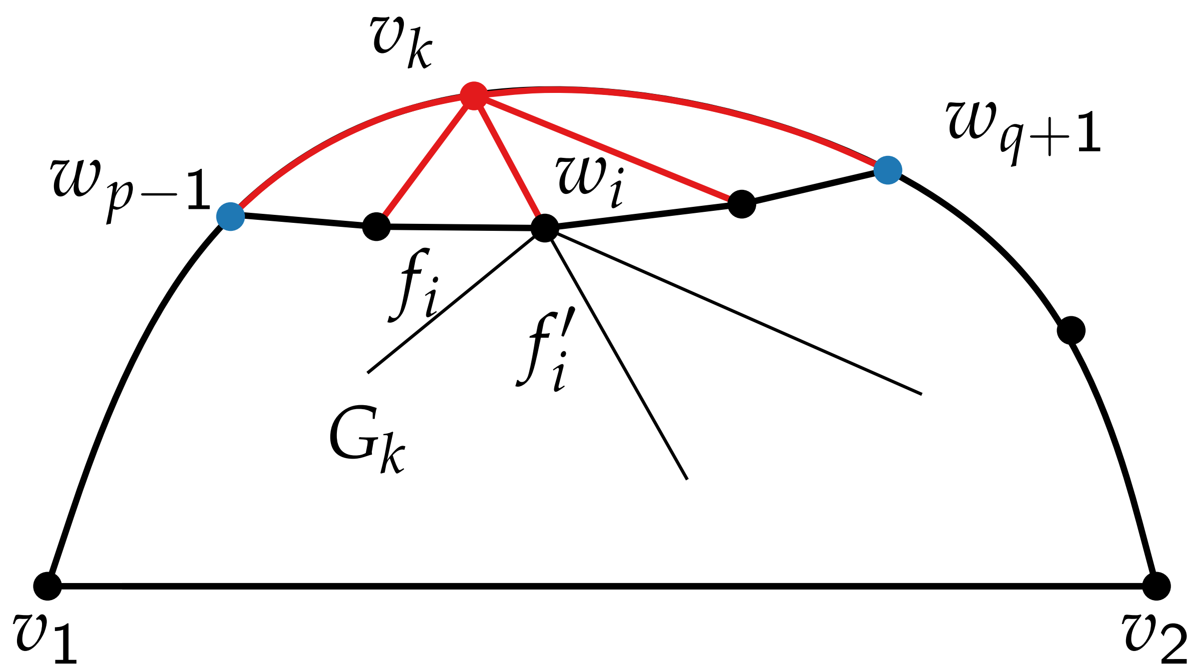


Canonical order – implementation

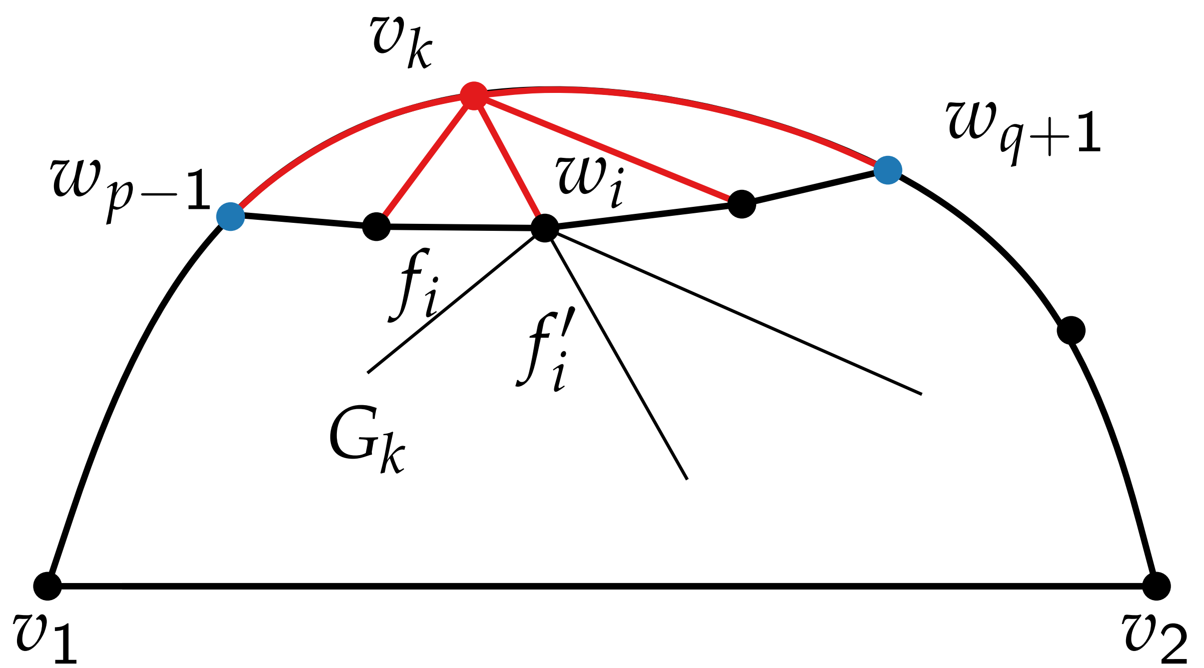
Remove v_k with sepF(v_k)$= 0$

- v_k and faces that contain v_k are removed
- outV(f_i) increases by two, $p+1 \leq i \leq q$
- outV(f_p), outV(f_{q+1}) increases by one
- outV(f_i') incrases by one, $p \leq i \leq q$
- outE(f_i) increases by one, $p \leq i \leq q+1$

- if $f_i$or f_i' becomes separating
  - increase sepF(u) by one for all its vertices u

- face f_i contains edge (w_{i-1}, w_i) of the outerface of G_{k-1}
- face f_i' contains edges of w_i that are in the interior of G_{k-1}

- f_{out} $=$ current outerface
- $F(v)$ $=$ faces that contain v
- $F(e)$ $=$ faces that contain e
- outV(f) $= \#$ vertices of f on f_{out}
- outE(f) $= \#$ edges of f on f_{out}
- sepF(v) $= \#$ separation faces that contain v



Canonical order – implementation

Algorithm CanonicalOrder

initialize;
for $k = n$ **to** 3 **do**
  choose $v_k \neq v_1, v_2$ such that
  – sepf(v)=0 or
  – or $F(v) = \{f\}$, outV(f)=3 and outE(f)=2

- f_{out} = current outerface
- $F(v)$ = faces that contain v
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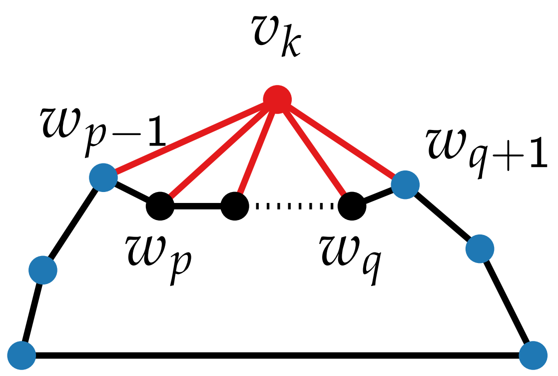

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replace v_k with path $P = w_p \dots w_q$ in f_{out};

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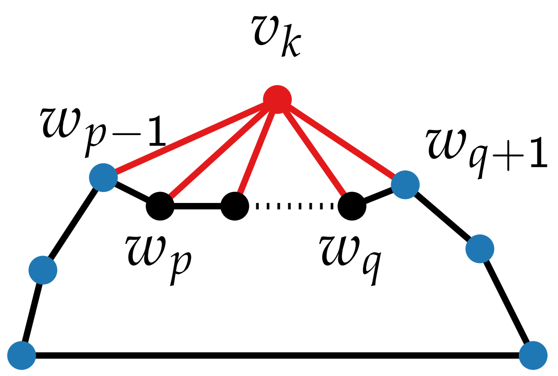

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    replace v_k with path $P = w_p \ldots w_q$ in f_{out};
    forall $p - 1 \leq i \leq q$ **do**
        remove face $\{v_k, w_i, w_{i+1}\}$ from $F(w_i)$and $F(w_{i+1})$;

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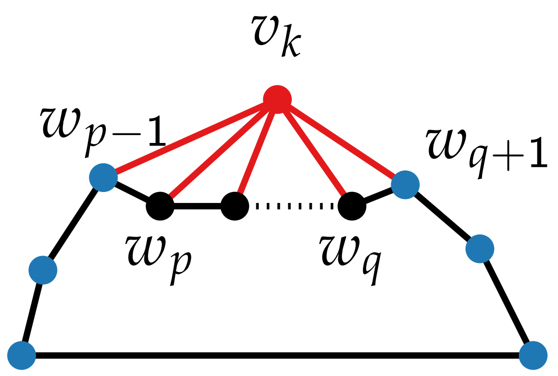

Canonical order – implementation

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  - – or $F(v) = \{f\}$, outV(f)=3 and outE(f)=2
- replace v_k with path $P = w_p \ldots w_q$ in f_{out};
- **forall** $p - 1 \leq i \leq q$ **do**
  - remove face $\{v_k, w_i, w_{i+1}\}$ from $F(w_i)$and $F(w_{i+1})$;
- **forall** $w \in w_{p-1}Pw_{q+1}$ **do**
  - **forall** $f \in F(w)$ **do**
    - update outV(f);
- **forall** $e \in w_{p-1}Pw_{q+1}$ **do**
  - **forall** $f \in F(e)$ **do**
    - update outE(f);

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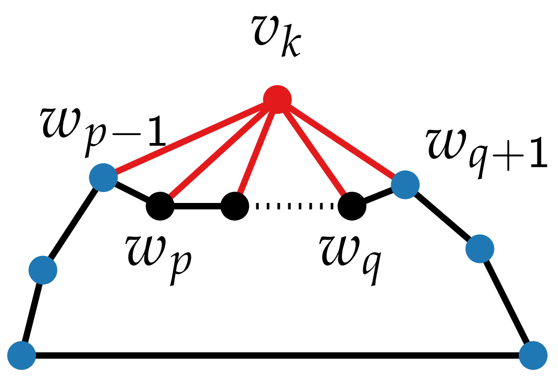

Canonical order – implementation

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    remove face $\{v_k, w_i, w_{i+1}\}$ from $F(w_i)$and $F(w_{i+1})$;
  forall $w \in w_{p-1}Pw_{q+1}$ **do**
    forall $f \in F(w)$ **do**
      update outV(f);
  forall $e \in w_{p-1}Pw_{q+1}$ **do**
    forall $f \in F(e)$ **do**
      update outE(f);
  forall $w \in P \cup N(P)$ **do**
    forall $f \in F(w)$ **do**
      update sepF(w);

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
- outE(f) = # edges of f on f_{out}
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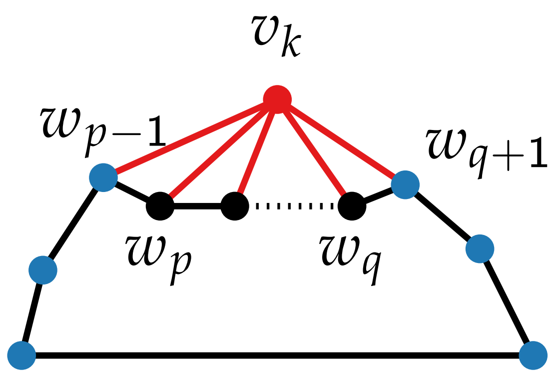

Canonical order – implementation

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- replace v_k with path $P = w_p \dots w_q$ in f_{out};
- **forall** $p - 1 \leq i \leq q$ **do**
  - remove face $\{v_k, w_i, w_{i+1}\}$ from $F(w_i)$and $F(w_{i+1})$;
- **forall** $w \in w_{p-1}Pw_{q+1}$ **do**
  - **forall** $f \in F(w)$ **do**
    - update outV(f);
- **forall** $e \in w_{p-1}Pw_{q+1}$ **do**
  - **forall** $f \in F(e)$ **do**
    - update outE(f);
- **forall** $w \in P \cup N(P)$ **do**
  - **forall** $f \in F(w)$ **do**
    - update sepF(w);

- f_{out} = current outerface
- $F(v)$ = faces that contain v
- $F(e)$ = faces that contain e
- outV(f) = # vertices of f on f_{out}
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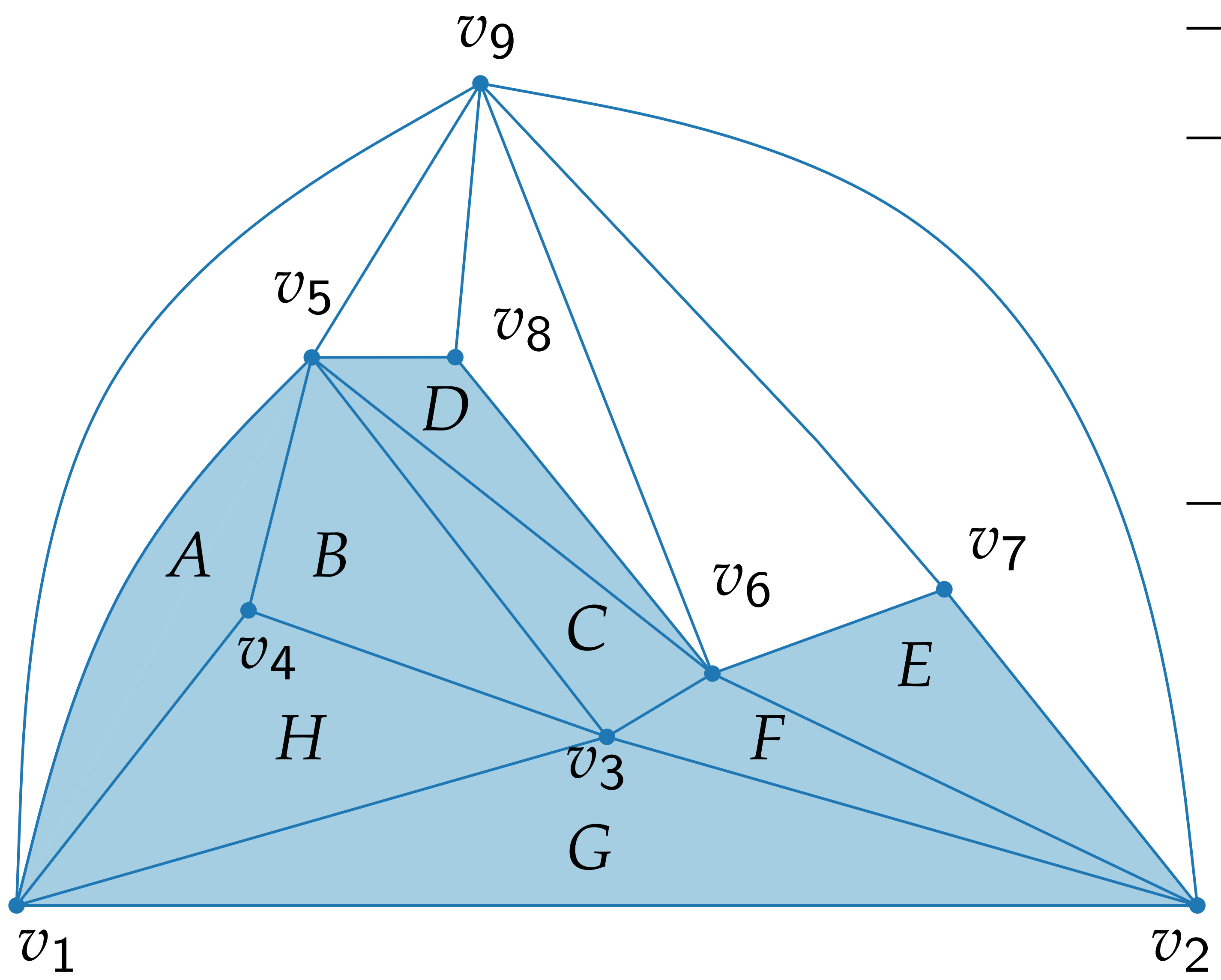
Lemma.
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

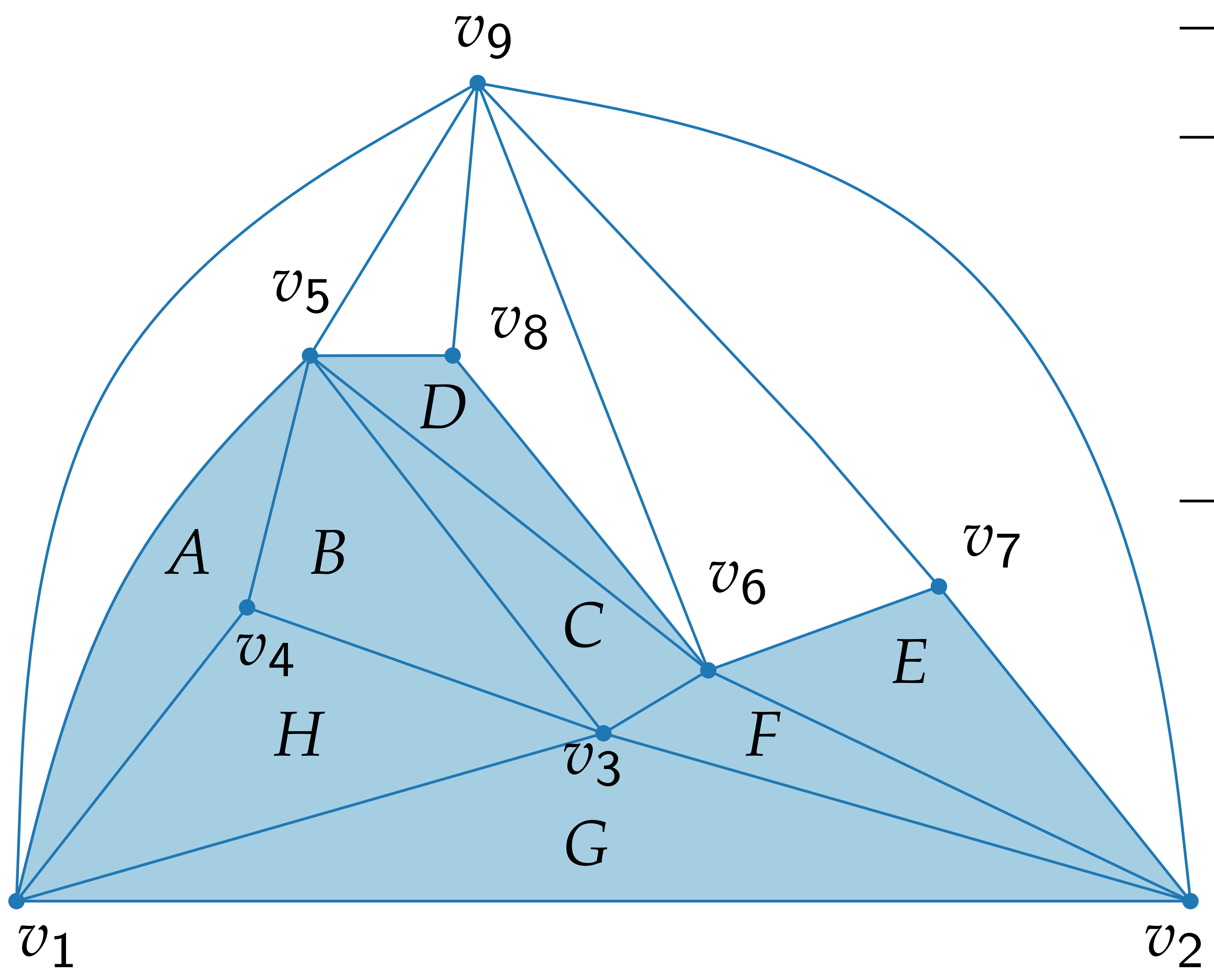
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)								
outE(f)								

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)						

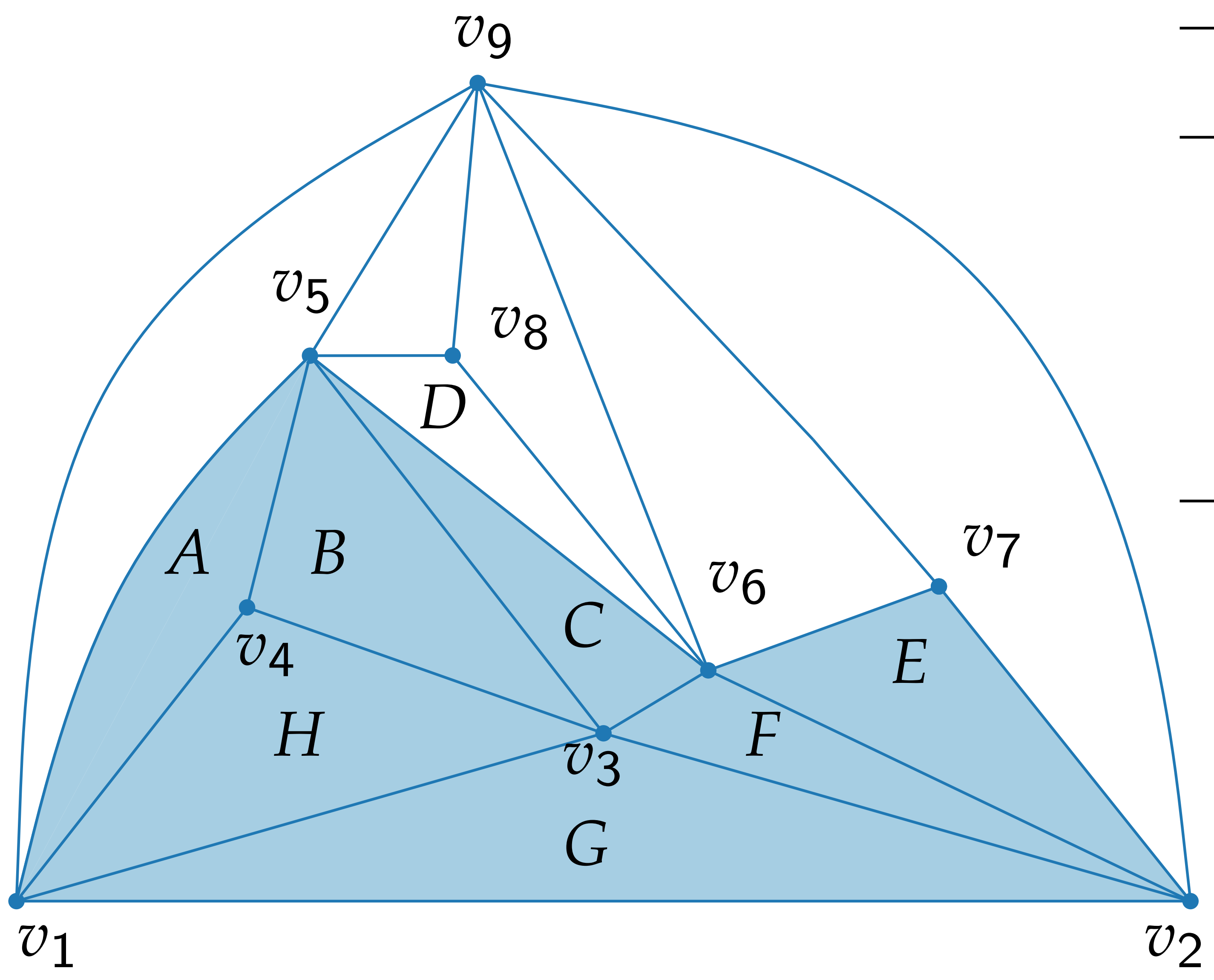
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	1	2	3	3	2	2	1
outE(f)	1	0	0	2	2	0	1	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)			2	4	1	1

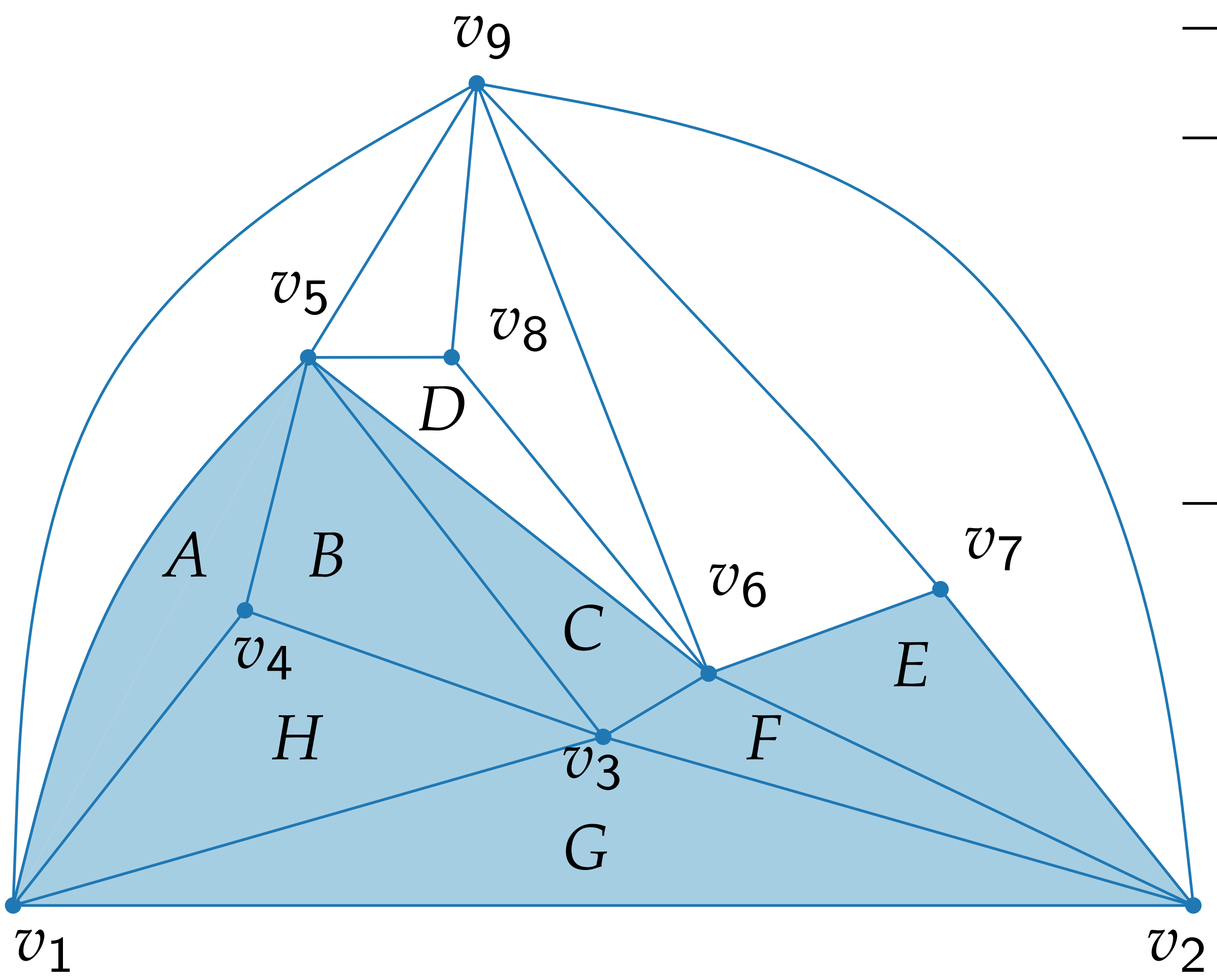
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	1	2	3	3	2	2	1
outE(f)	1	0	0	2	2	0	1	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)			2	4	1	1

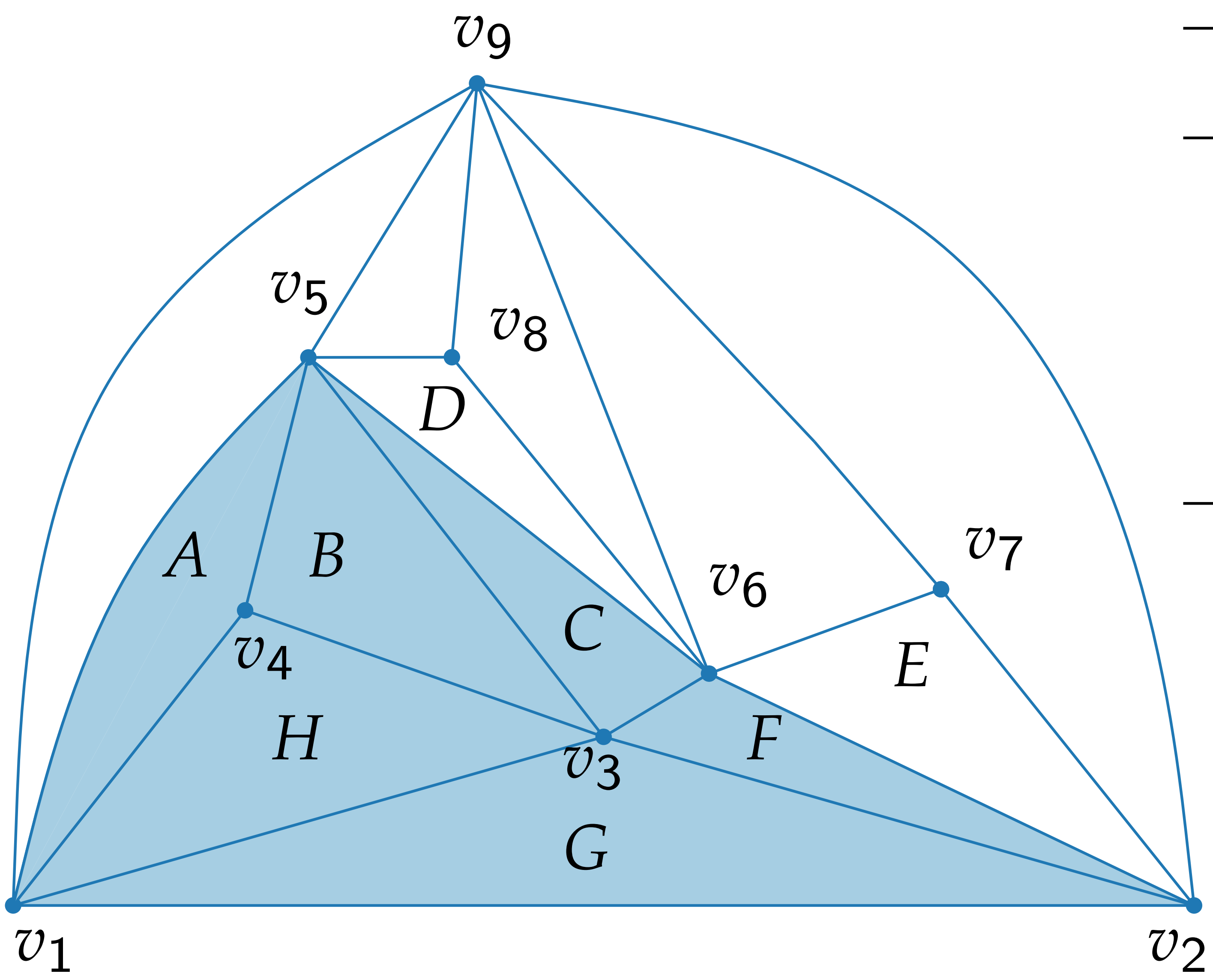
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	1	2		3	2	2	1
outE(f)	1	0	1		2	0	1	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)			0	2	1	

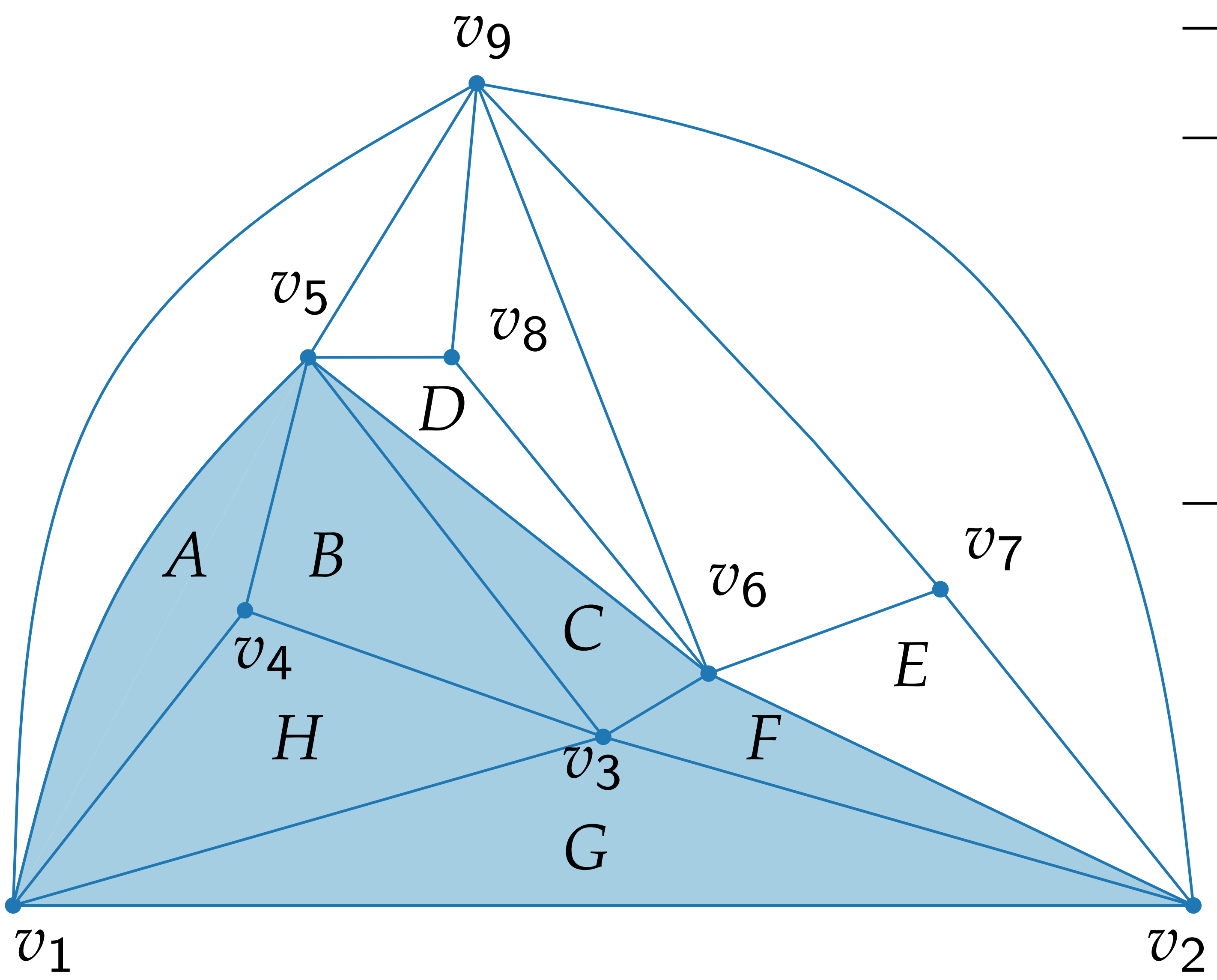
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	1	2		3	2	2	1
outE(f)	1	0	1		2	0	1	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)			0	2	1	

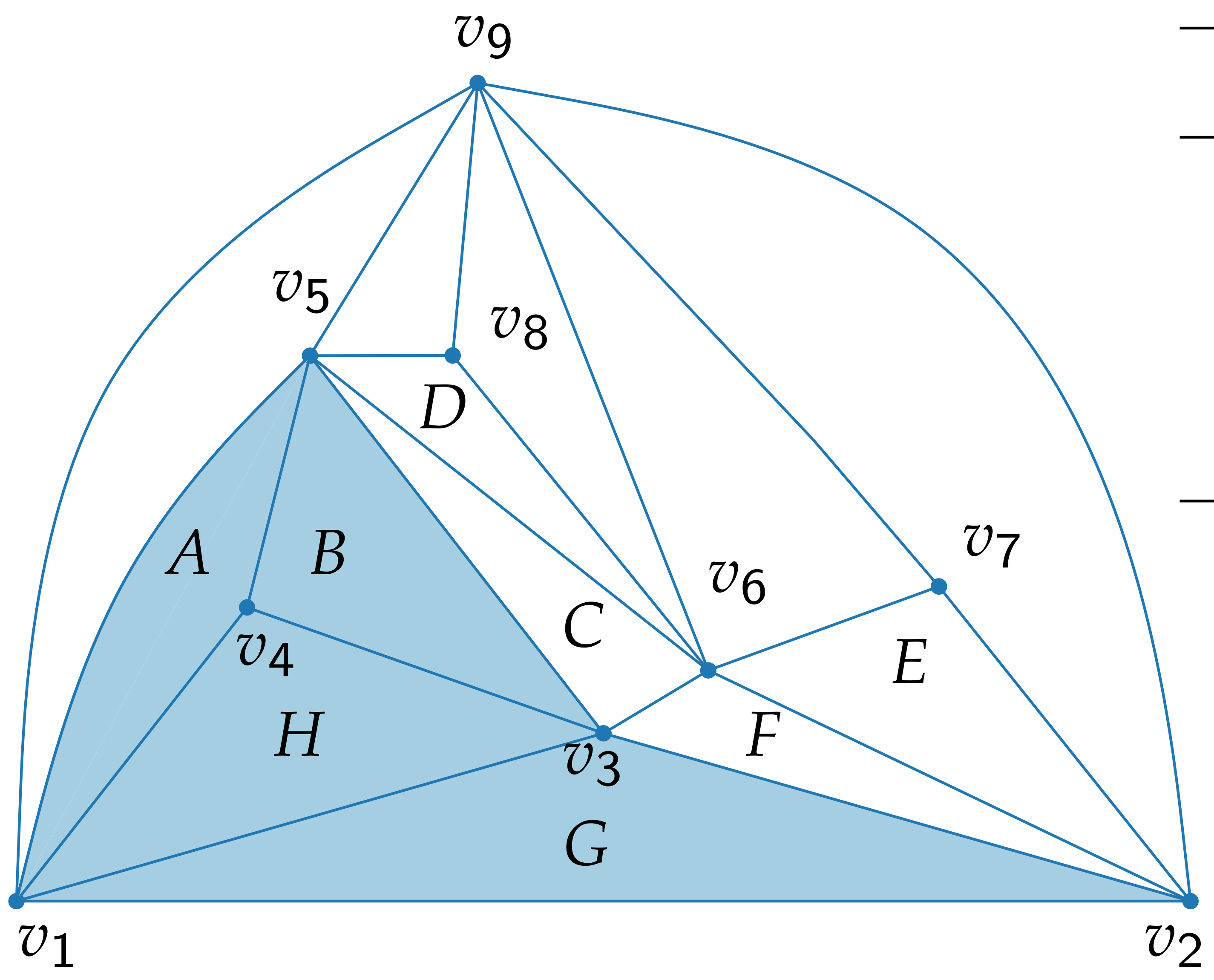
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	1	2			2	2	1
outE(f)	1	0	1			1	1	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)			0	0		

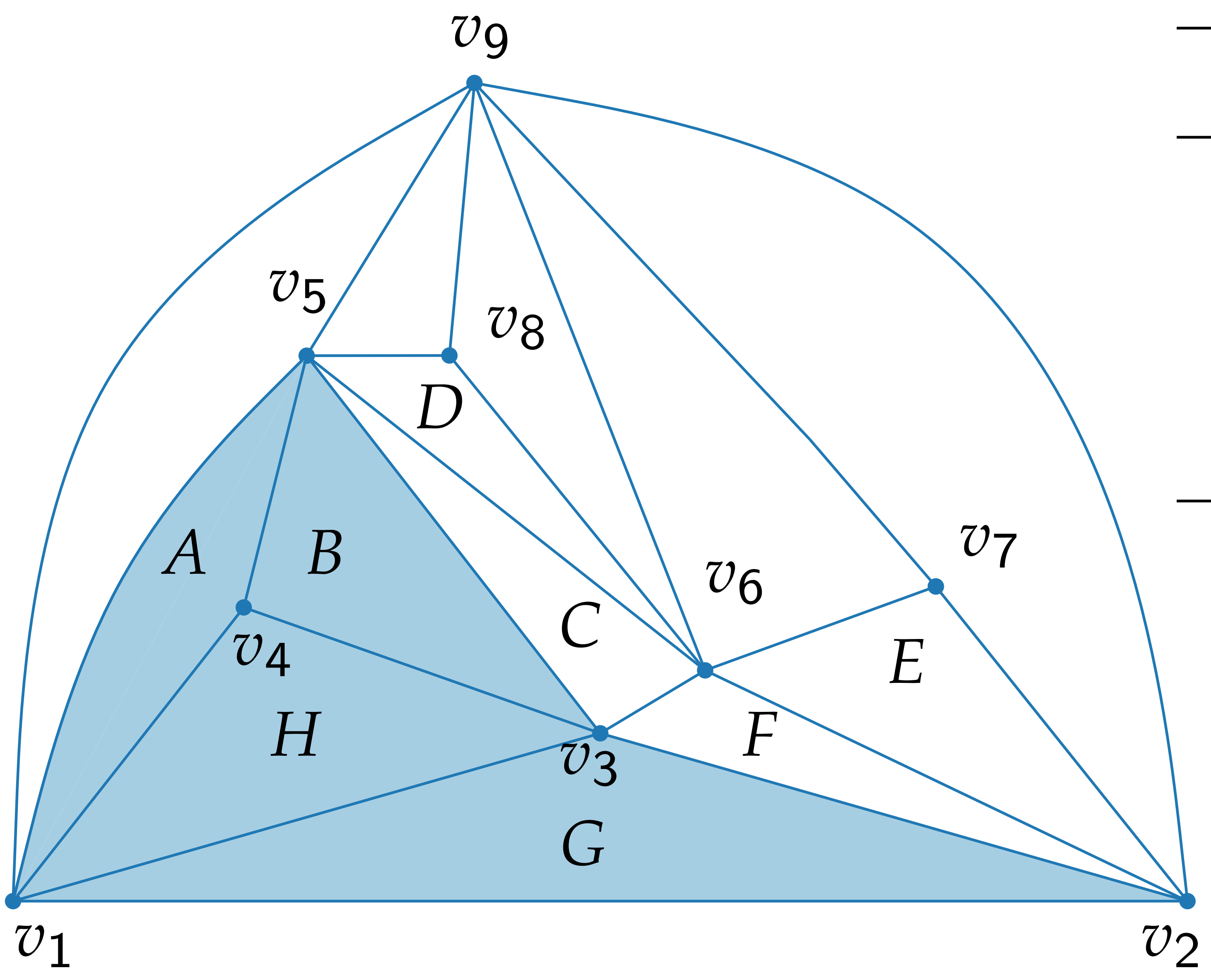
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	1	2			2	2	1
outE(f)	1	0	1			1	1	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)			0	0		

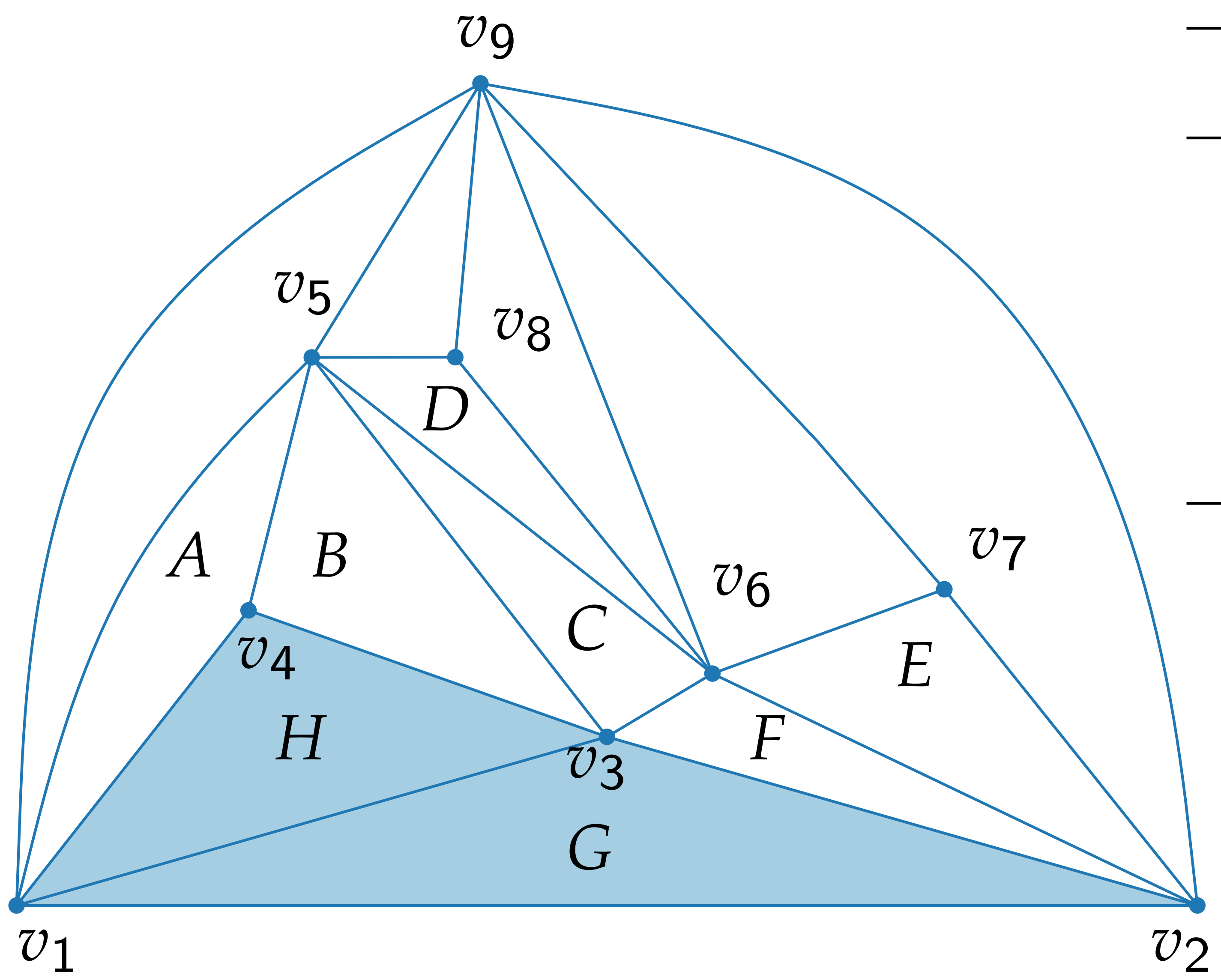
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	2					3	2
outE(f)	1	1					2	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)	2		0			

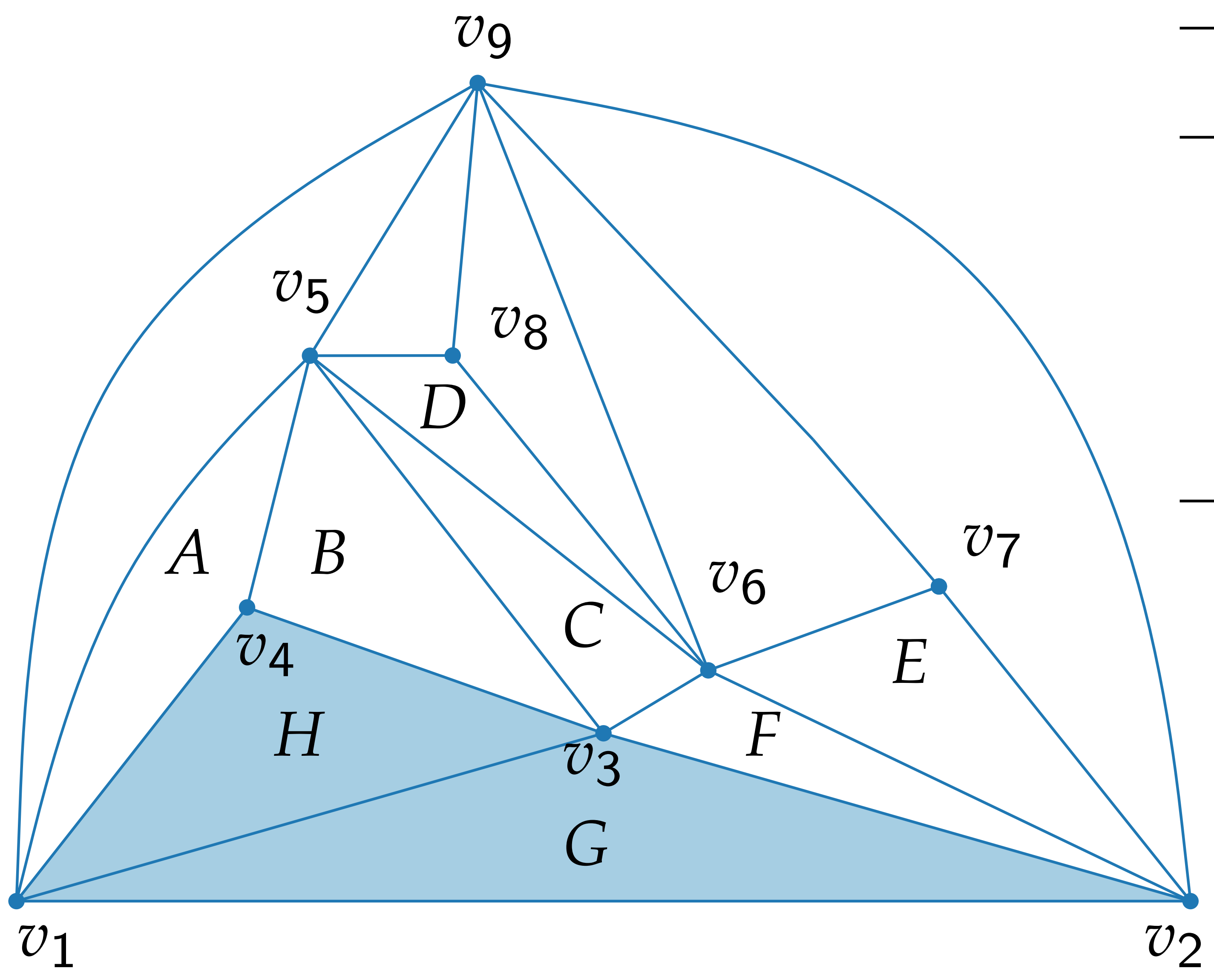
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)	2	2					3	2
outE(f)	1	1					2	0

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)	2		0			

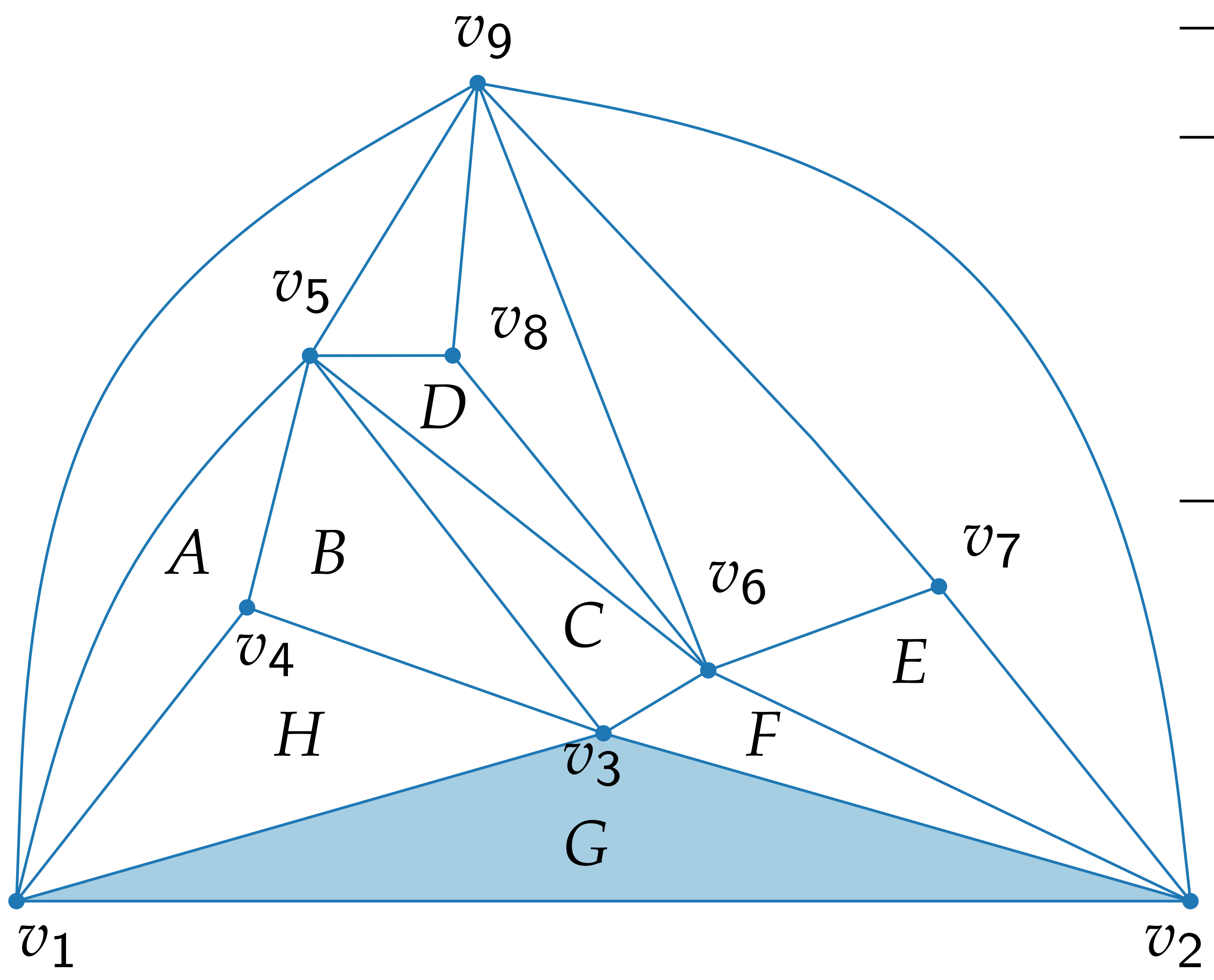
Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)							3	3
outE(f)							2	2

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)	2	1				

Canonical order – example



	A	B	C	D	E	F	G	H
outV(f)							3	3
outE(f)							2	2

	v_3	v_4	v_5	v_6	v_7	v_8
sepF(v)	2	1				

Order:

$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$

Literature

- [HGD Ch. 6.5] canonical order
- [dFPP90] de Fraysseix, Pach, Pollack *"How to draw a planar graph on a grid"*, Combinatorica, 1990
- [Kant96] Kant *"Drawing planar graphs using the canonical ordering"*, Algorithmica, 1996
- [BBC11] Badent, Brandes, Cornelsen *"More Canonical Ordering"*, JGAA, 2011