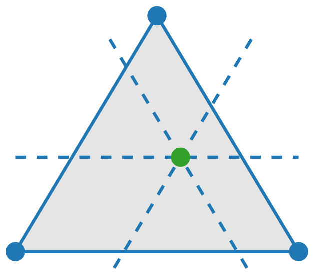
Visualisation of graphs

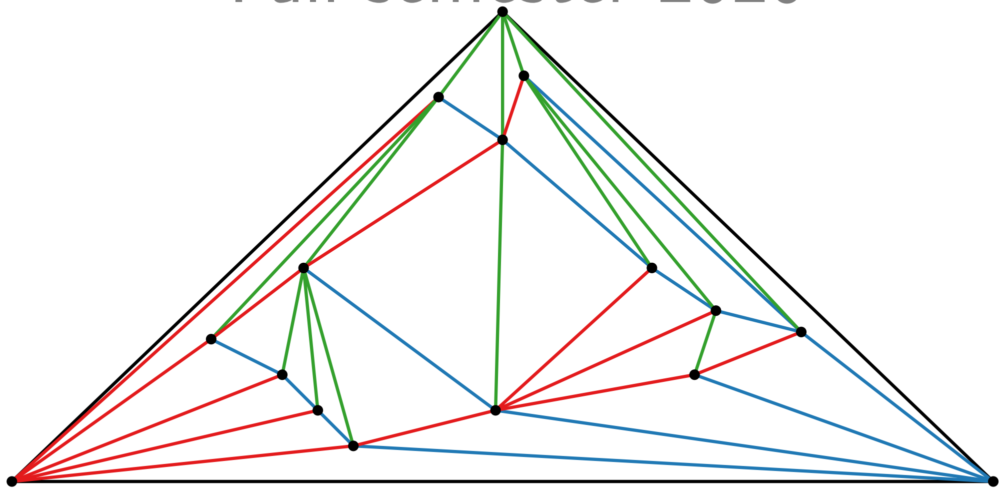
Planar straight-line drawings

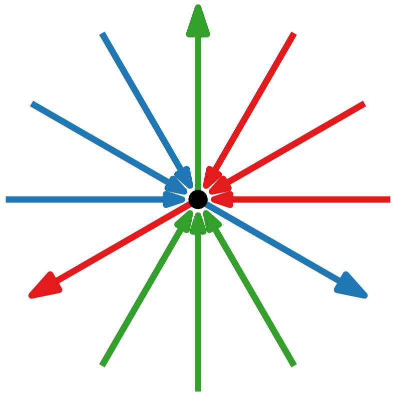
Schnyder realiser

Antonios Symvonis · Chrysanthi Raftopoulou

Fall semester 2020







The original slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, ...
The original presentation was modified/updated by A. Symvonis and C. Raftopoulou

Planar straight-line drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

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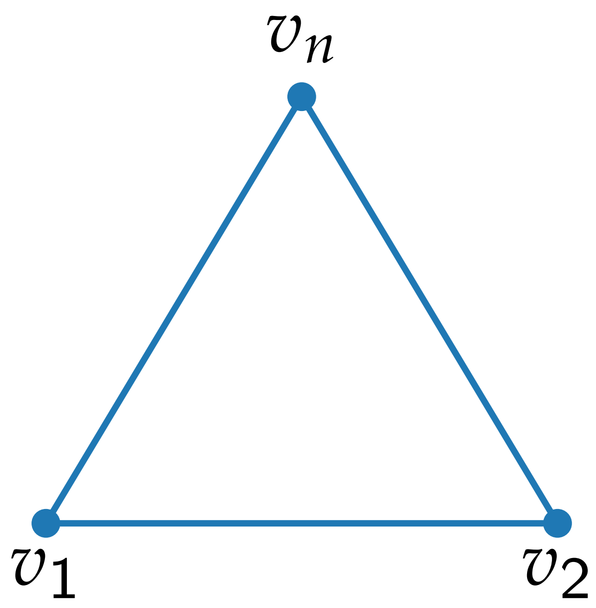

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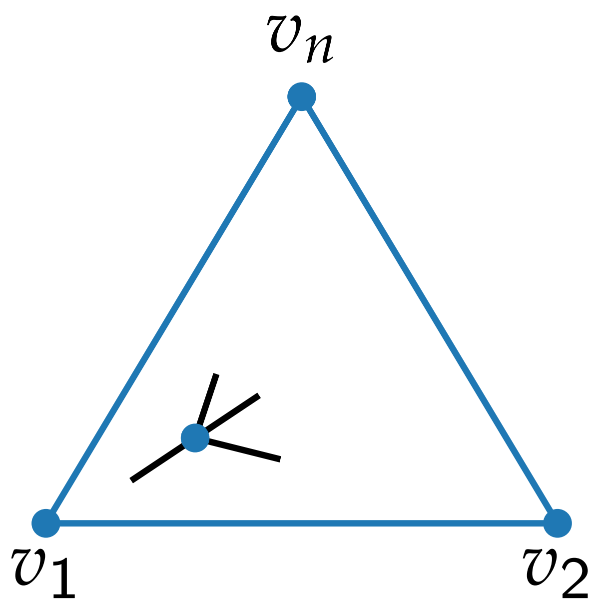

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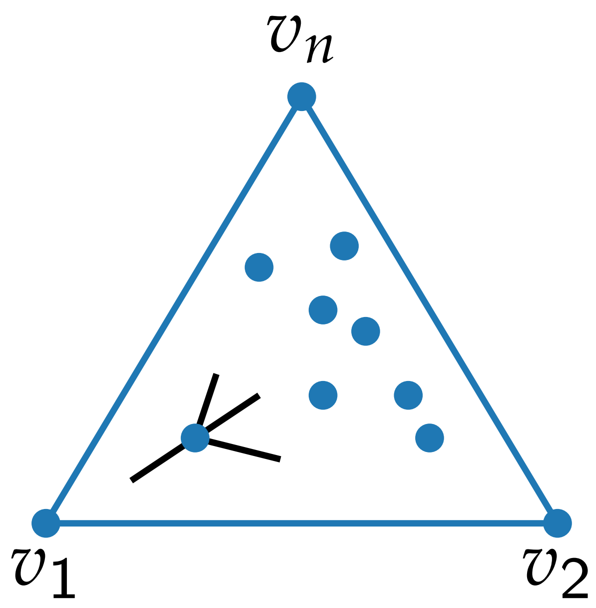

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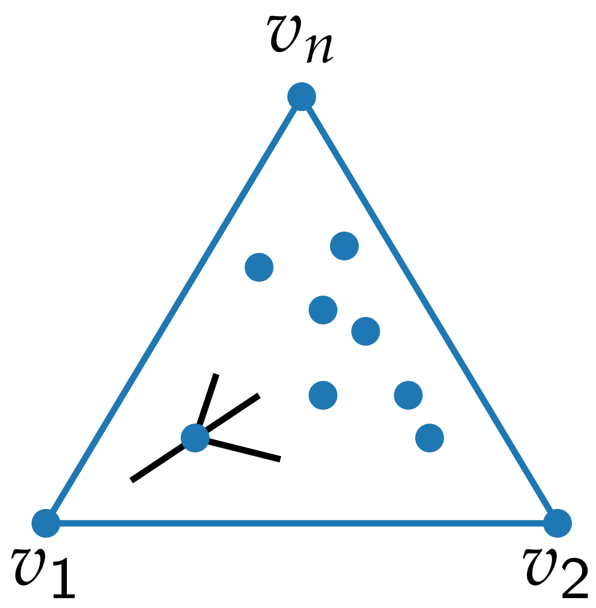

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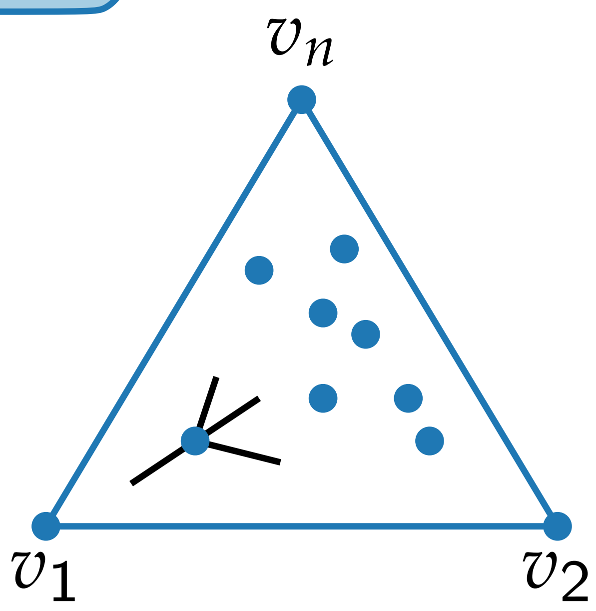

Barycentric coordinates

Definition.
Let $A, B, C, P \in \mathbb{R}^2$.
The **barycentric coordinates** of P with respect to $\triangle ABC$ are a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3_{\geq 0}$ such that

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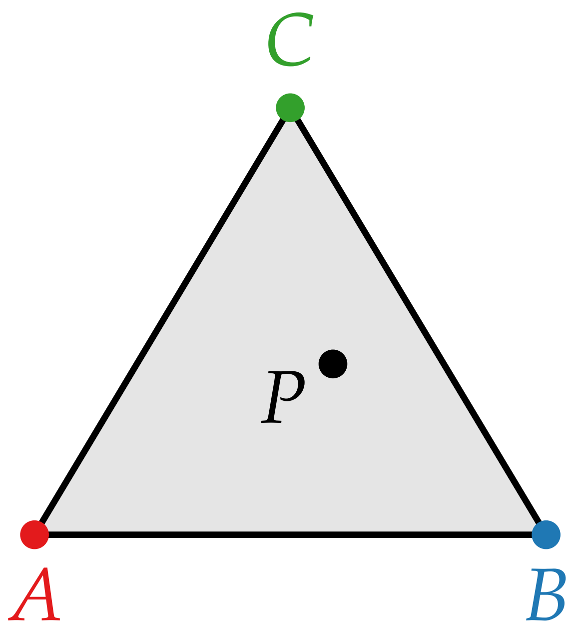

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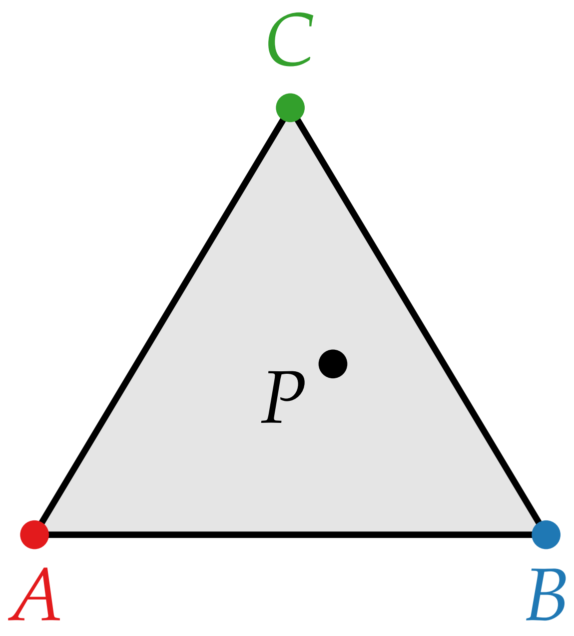

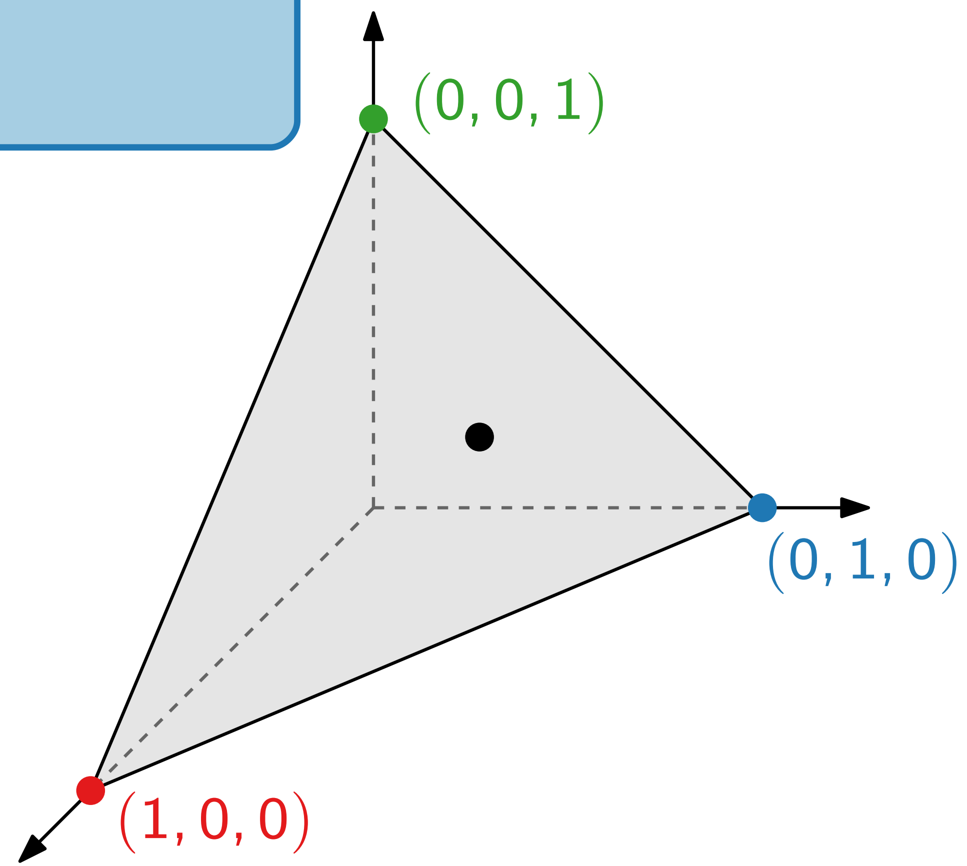

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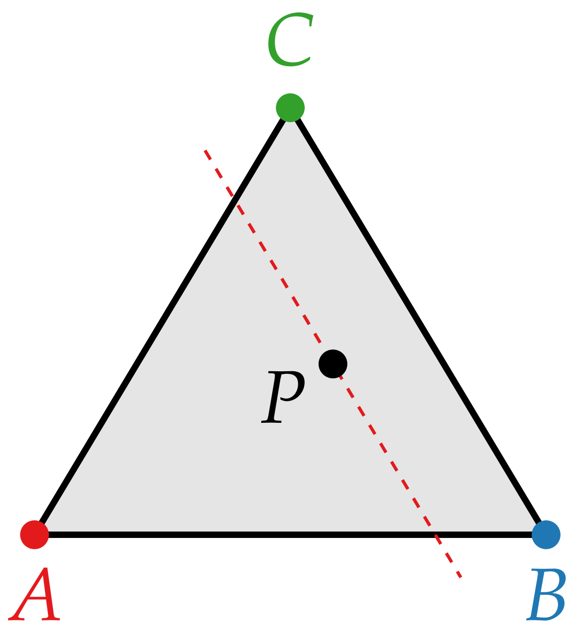

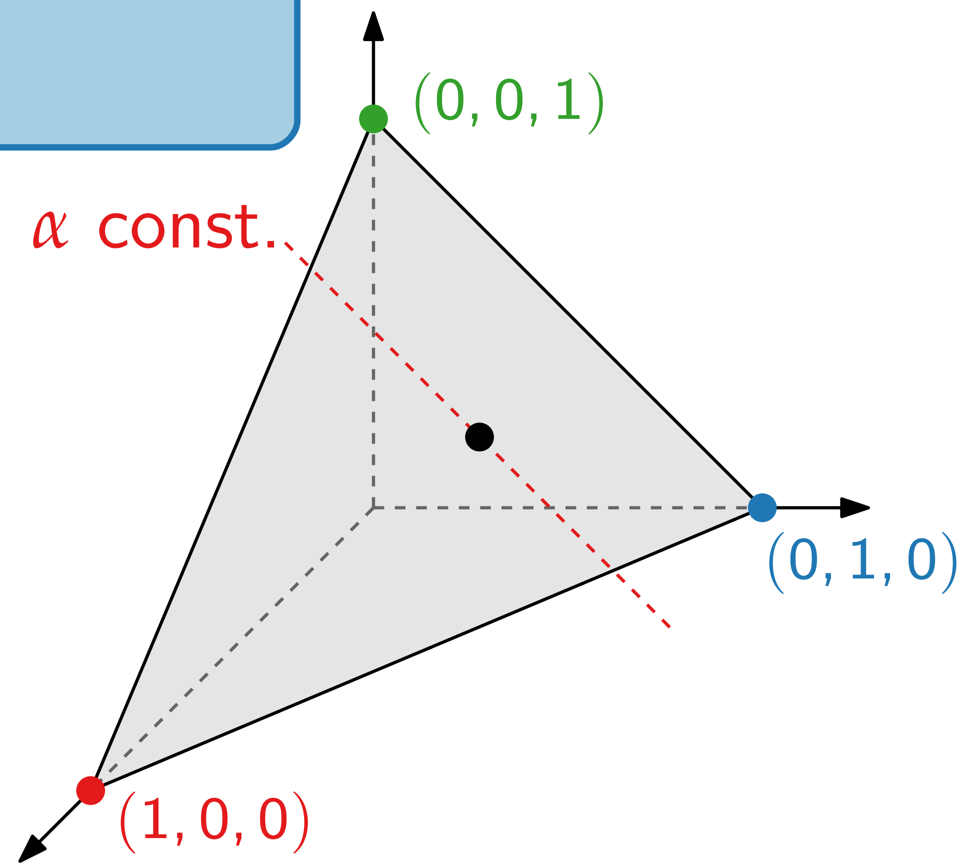

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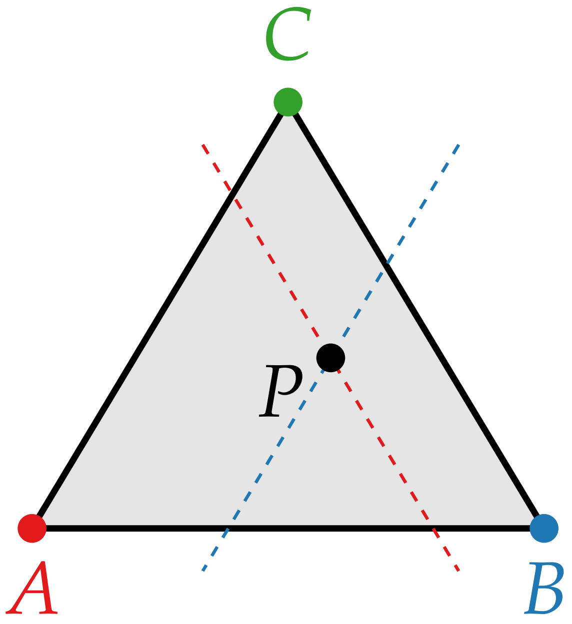

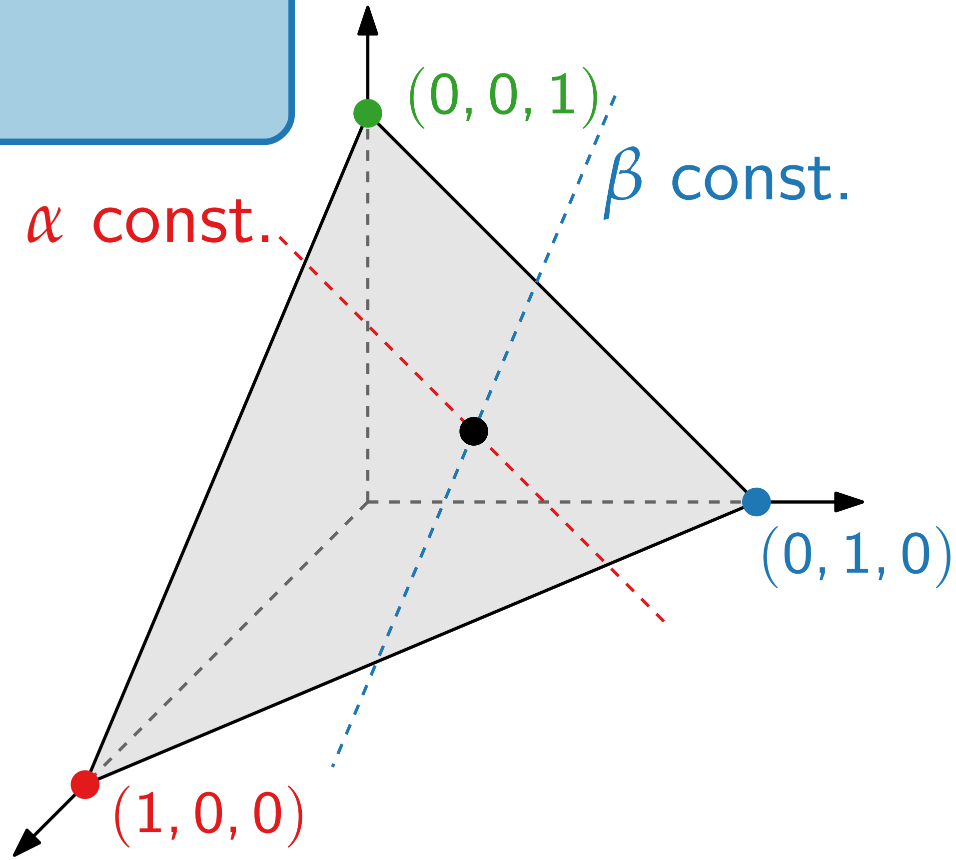

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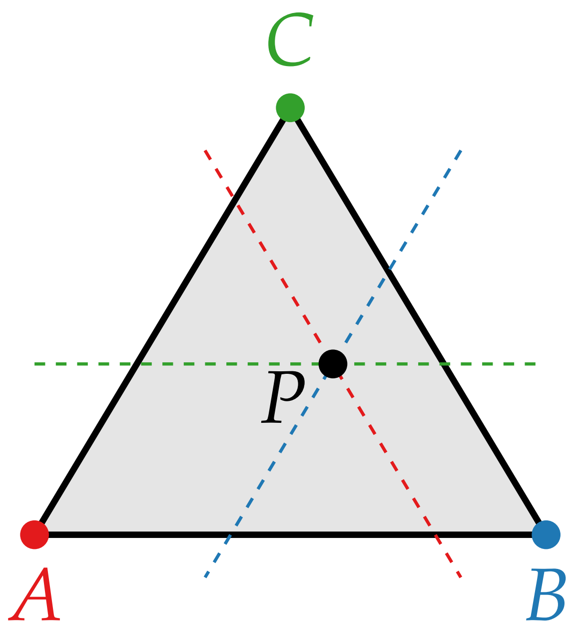

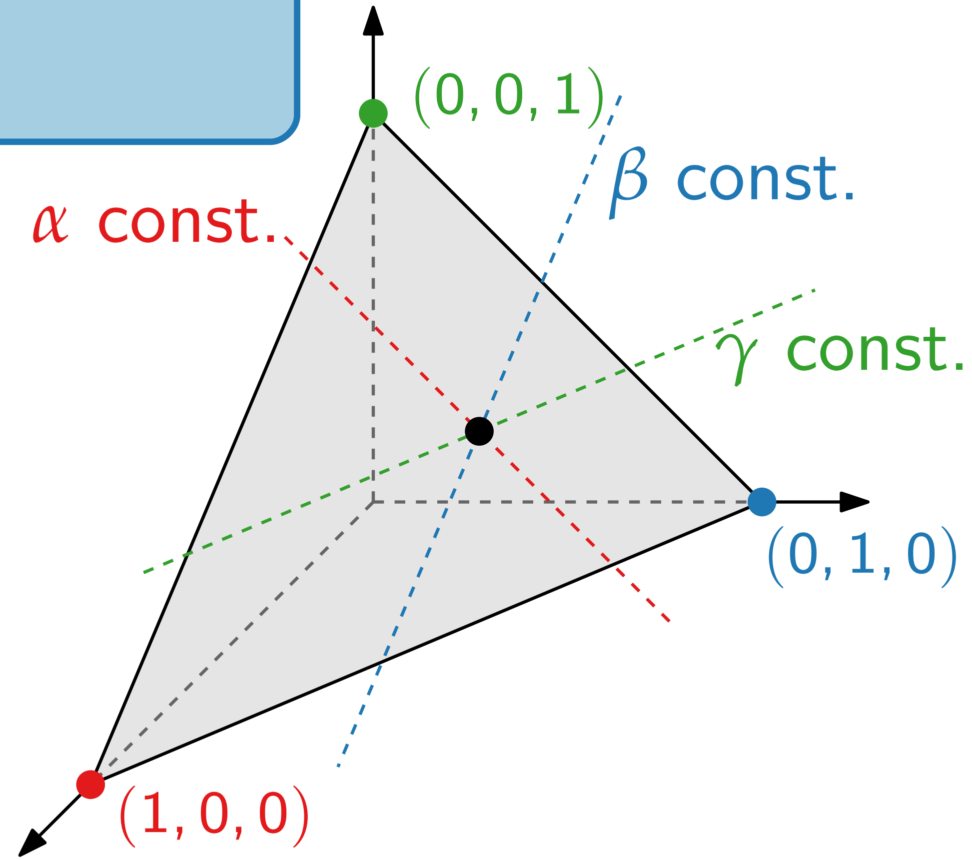

Barycentric representation

Definition.
A **barycentric representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of G; i.e. it is *injective* map $\phi \colon V \to \mathbb{R}^3_{\geq 0}$, $v \mapsto (v_1, v_2, v_3)$ with the following properties:

- $v_1 + v_2 + v_3 = 1$ for all $v \in V$
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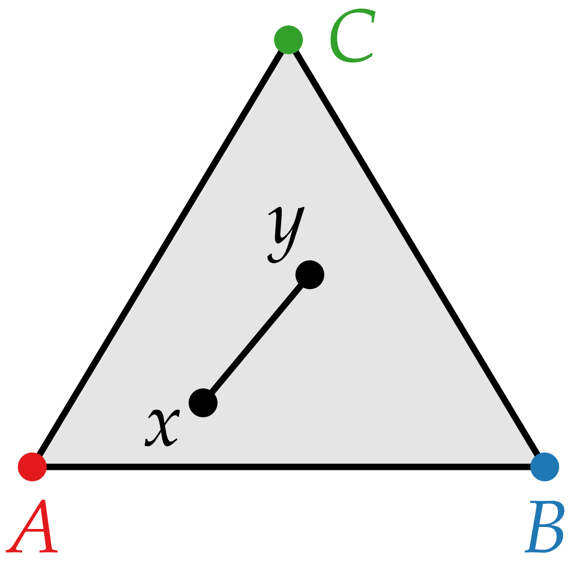

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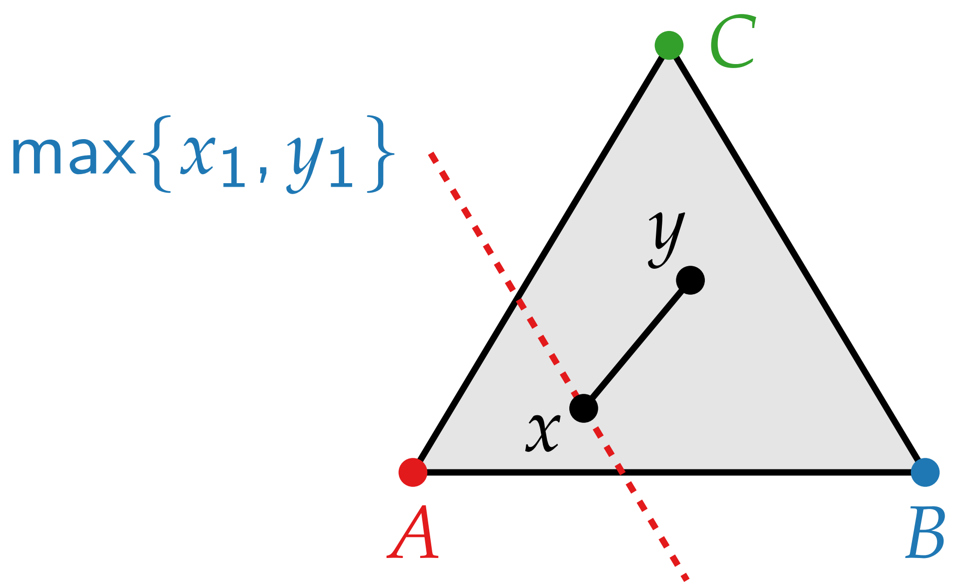

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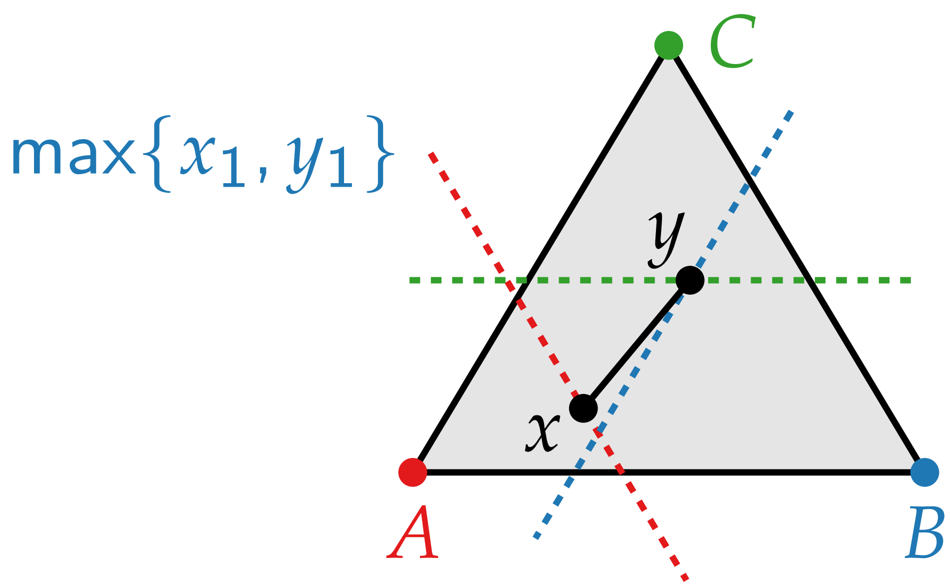

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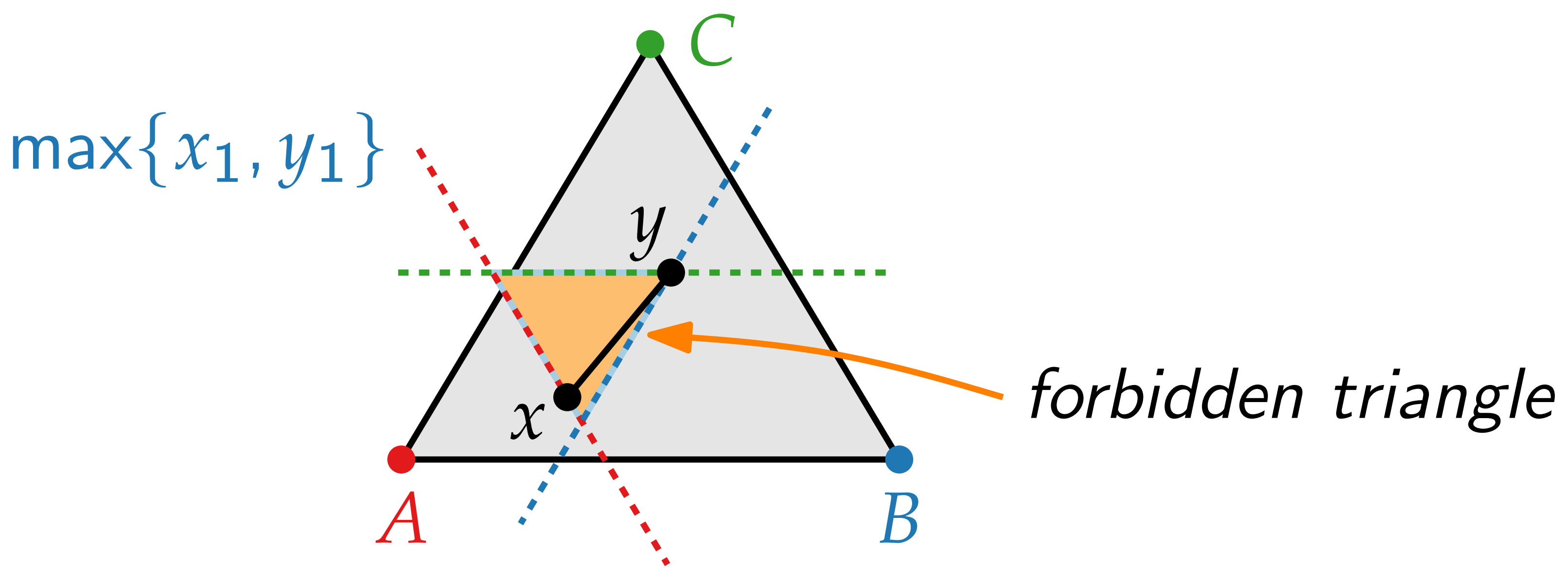

Barycentric representations & planar graphs

Lemma.
Let $\phi : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a graph $G = (V, E)$ and let $A, B, C \in \mathbb{R}^2$ in general position. Then the mapping

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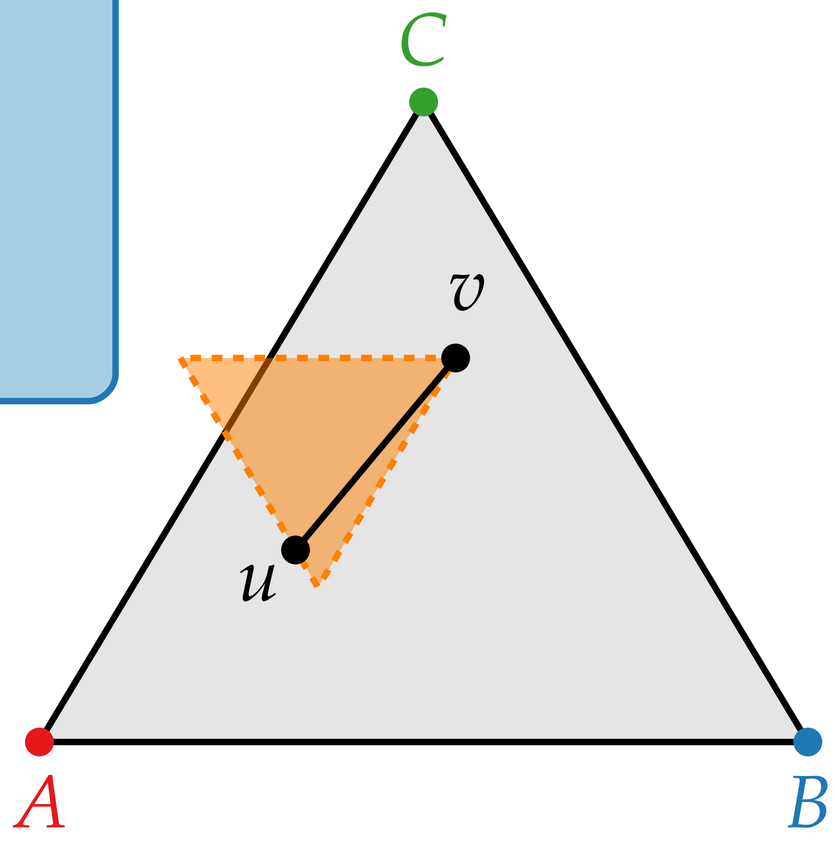

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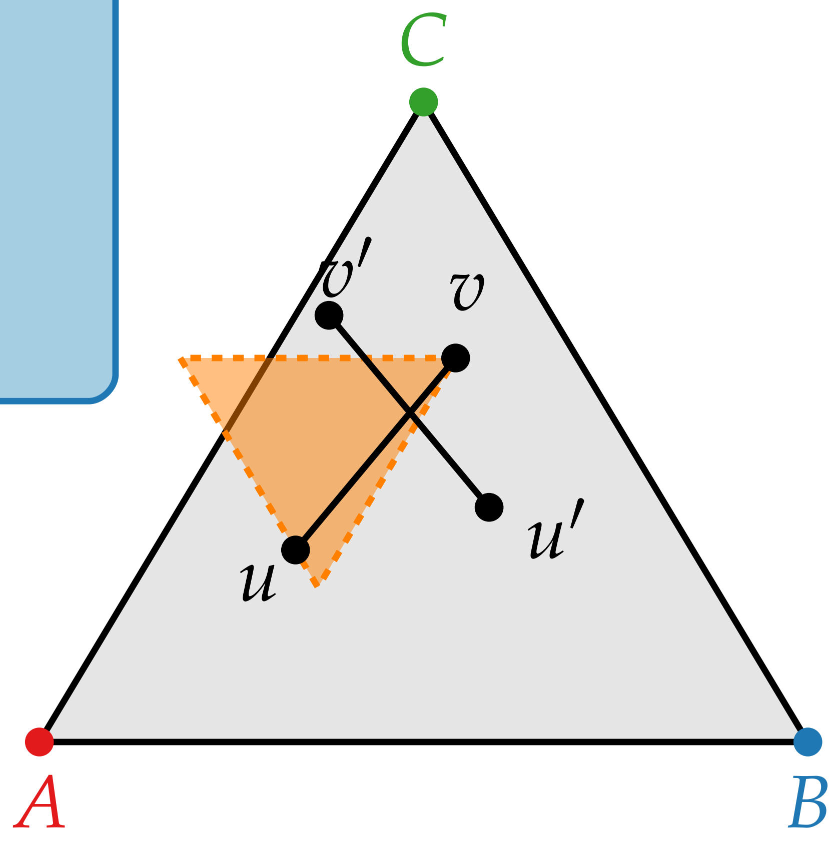

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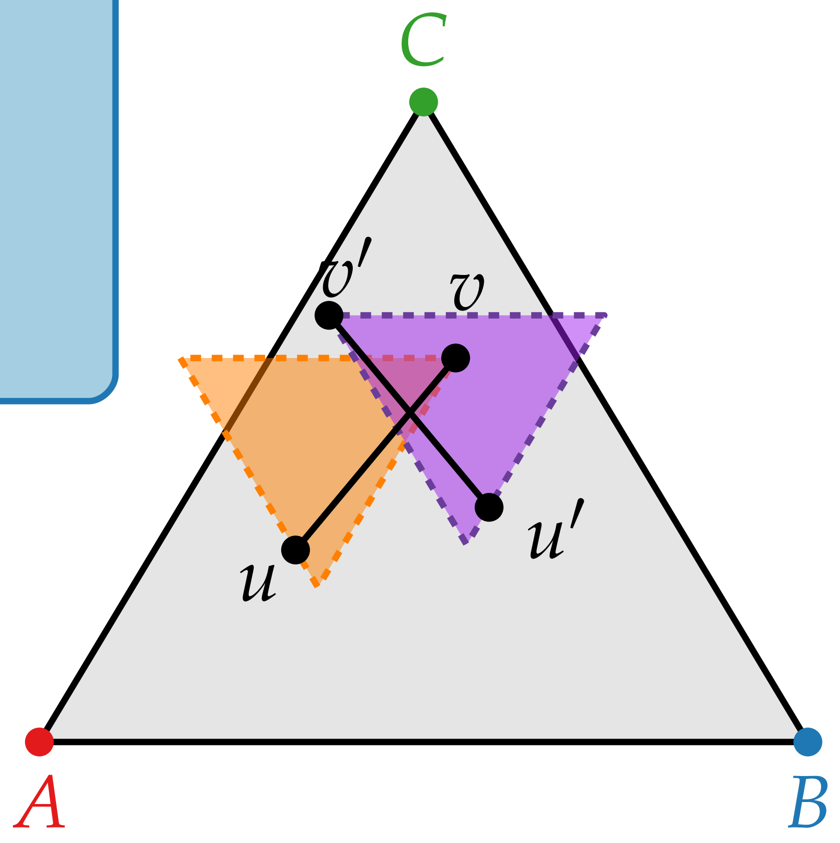

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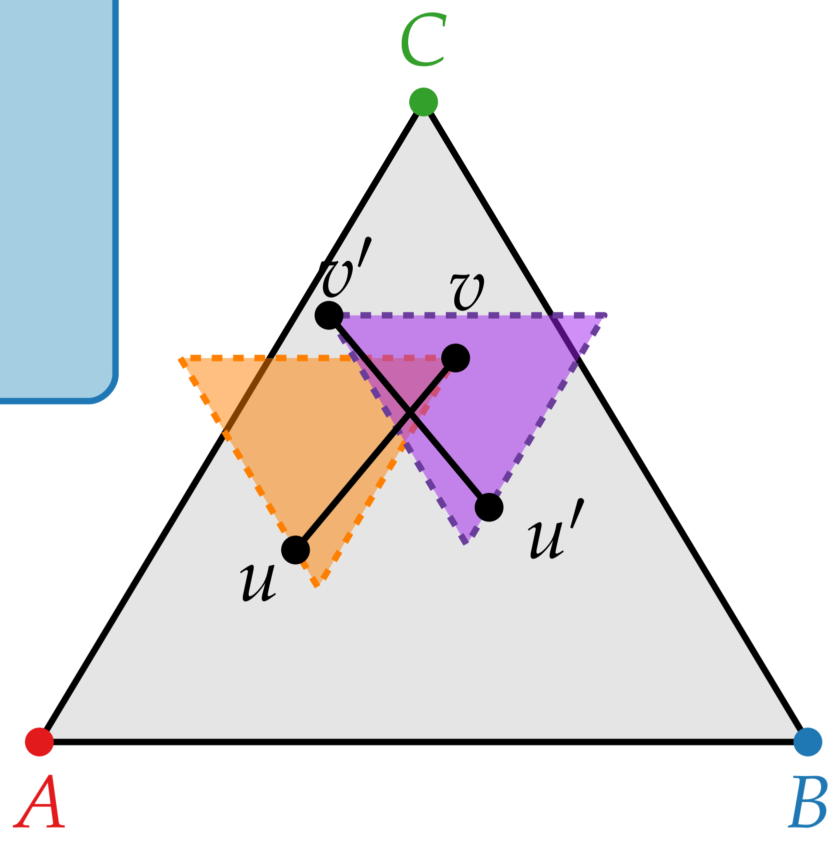

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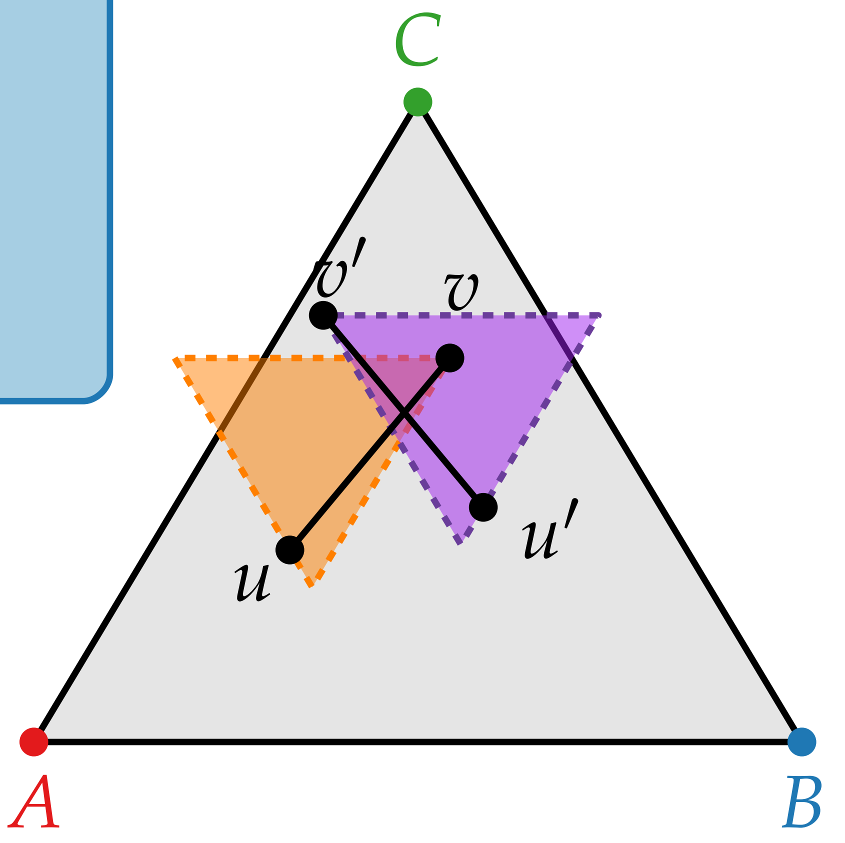

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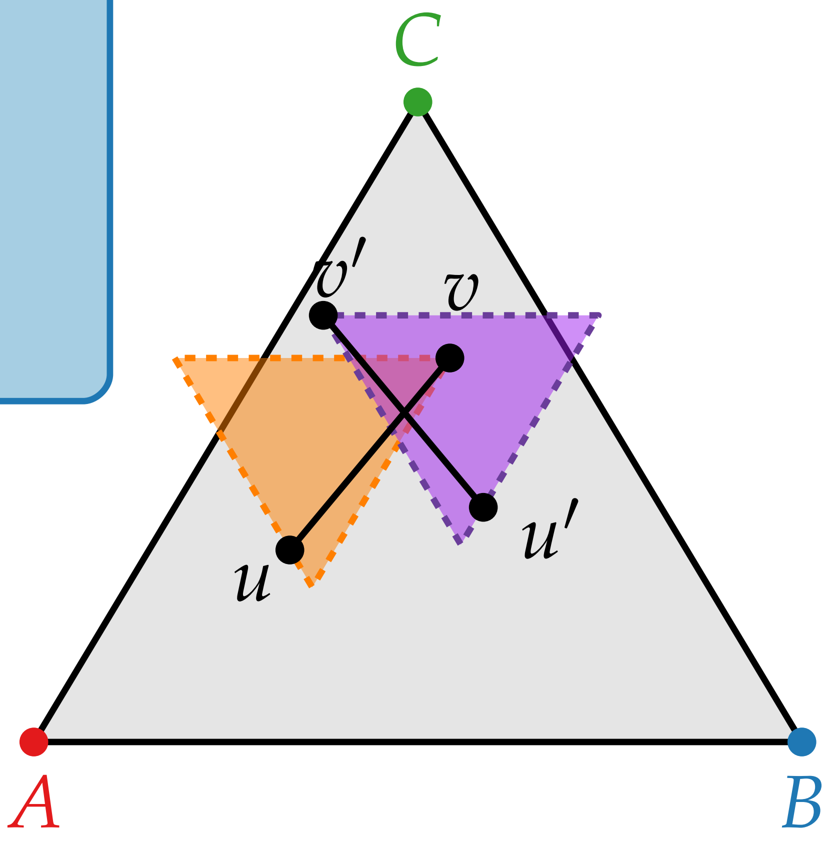

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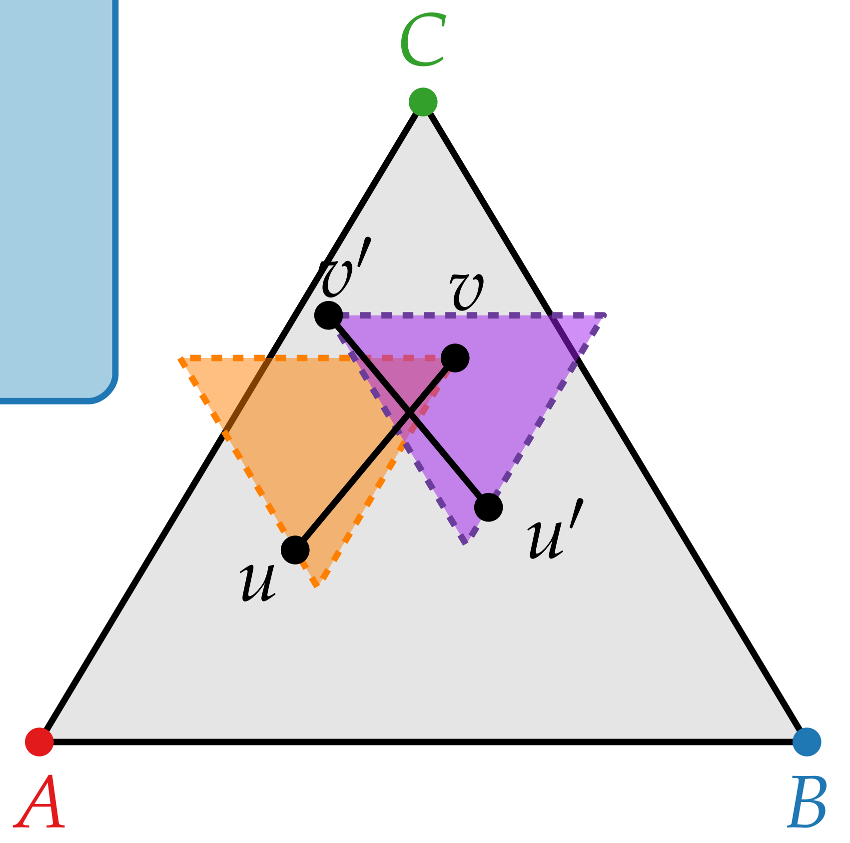
How to get vertices on grid?



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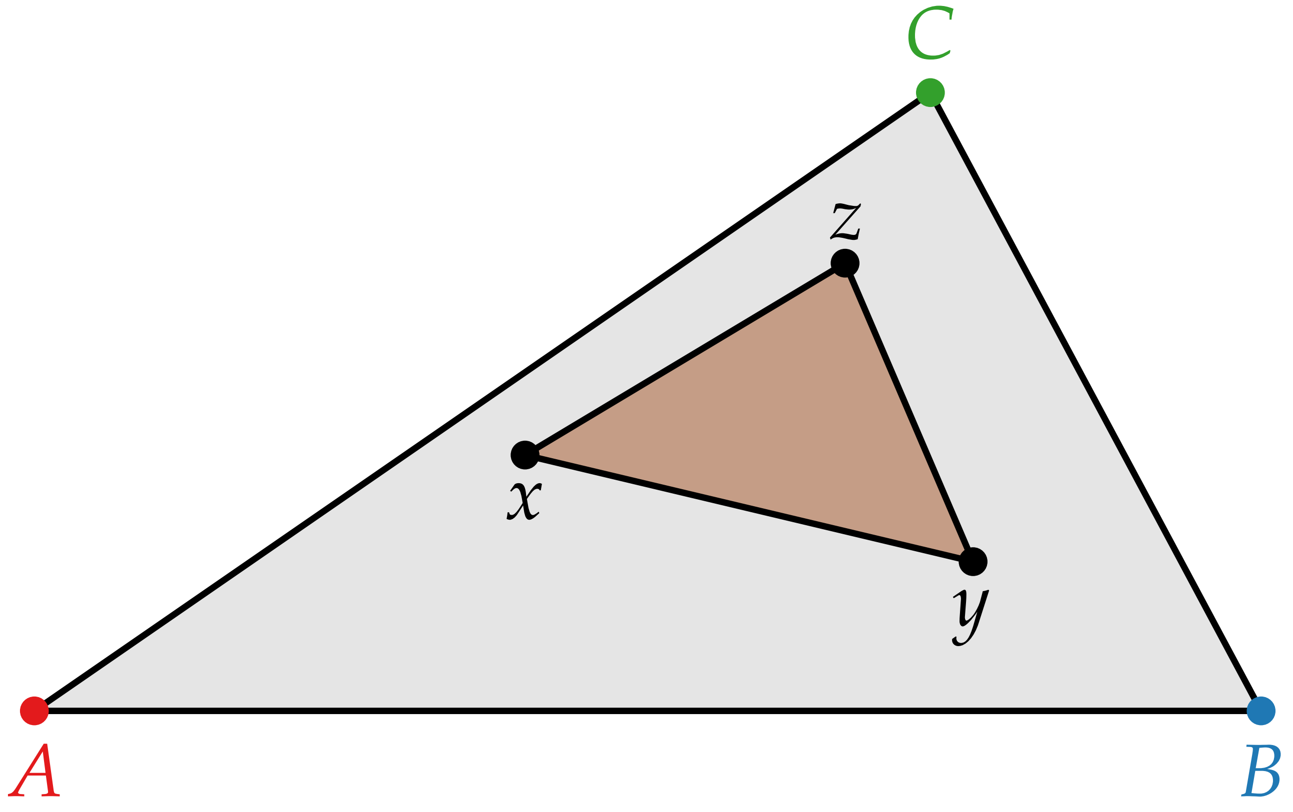
Angle labeling

Observation 1.
Let $v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a triangulated plane graph $G = (V, E)$.
We can **uniquely** label each angle $\angle(xy, xz)$ with $k \in \{1, 2, 3\}$.



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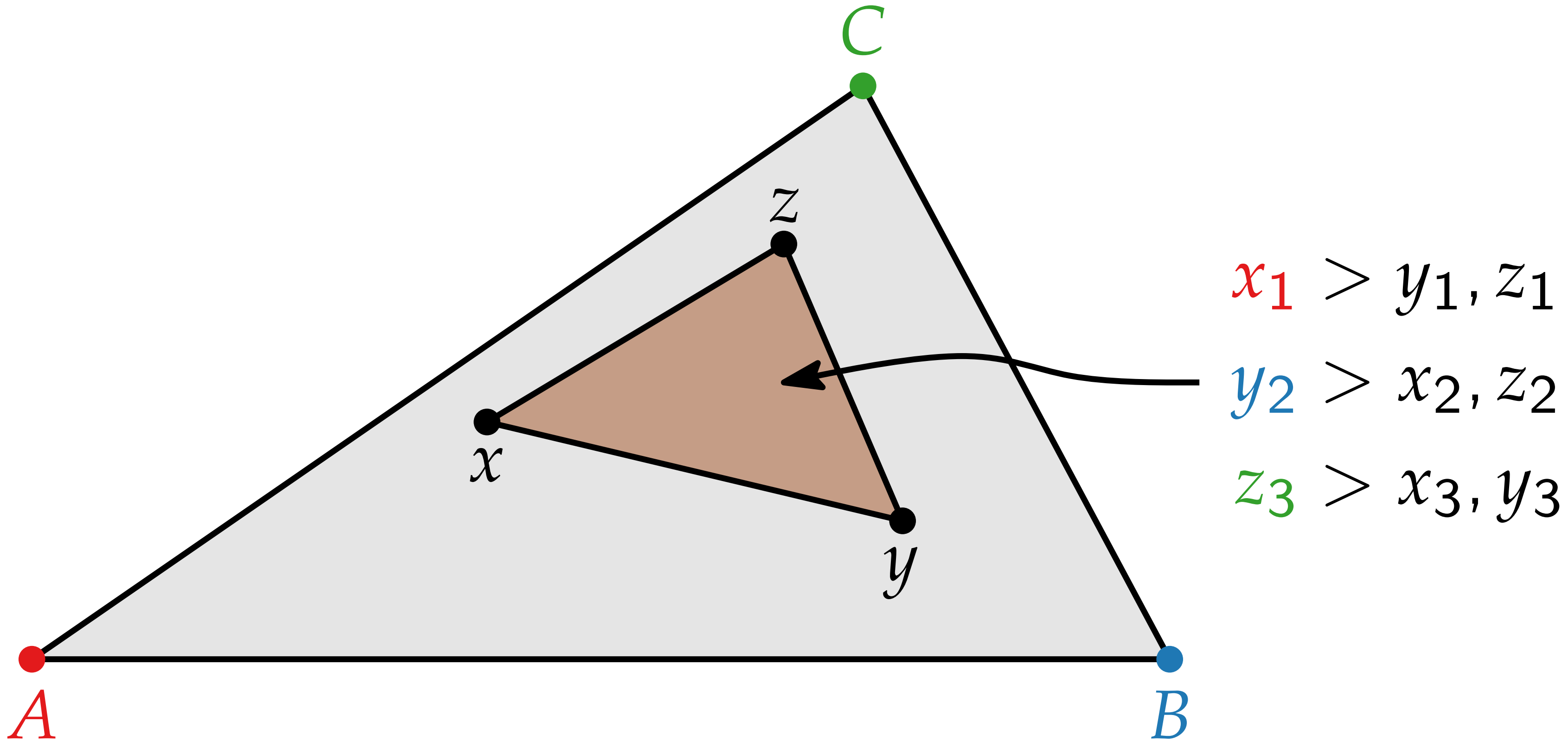

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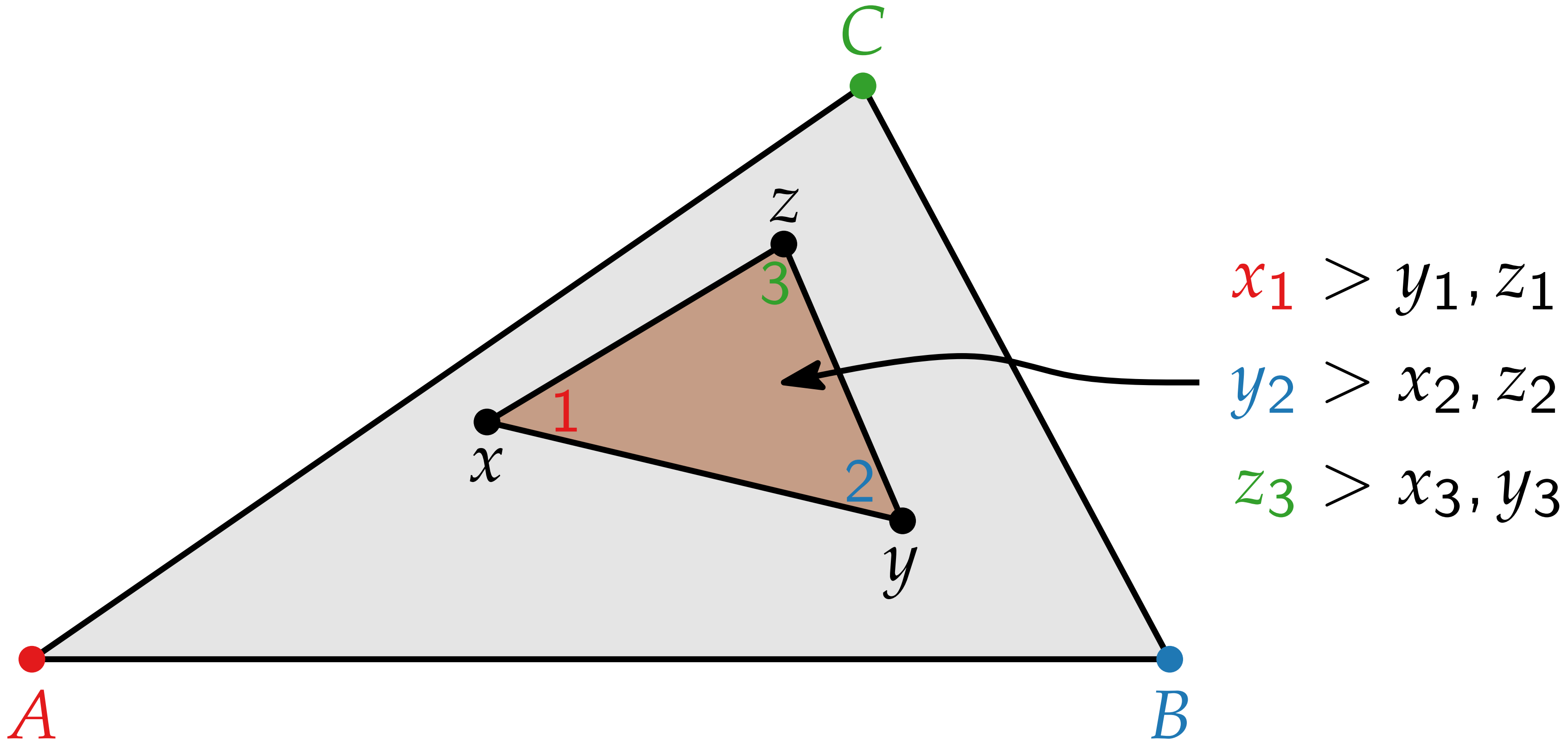

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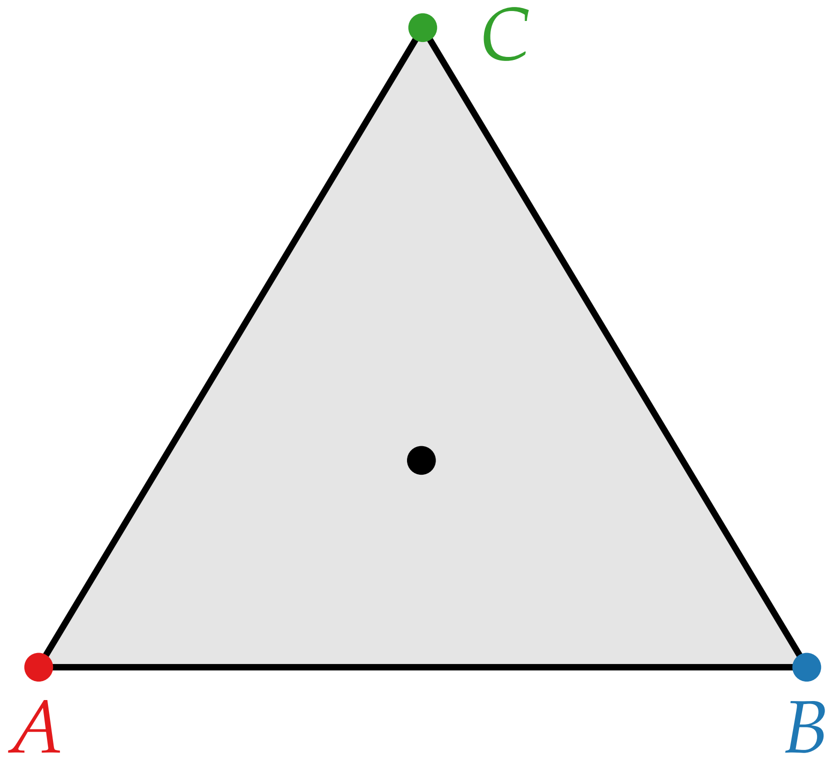

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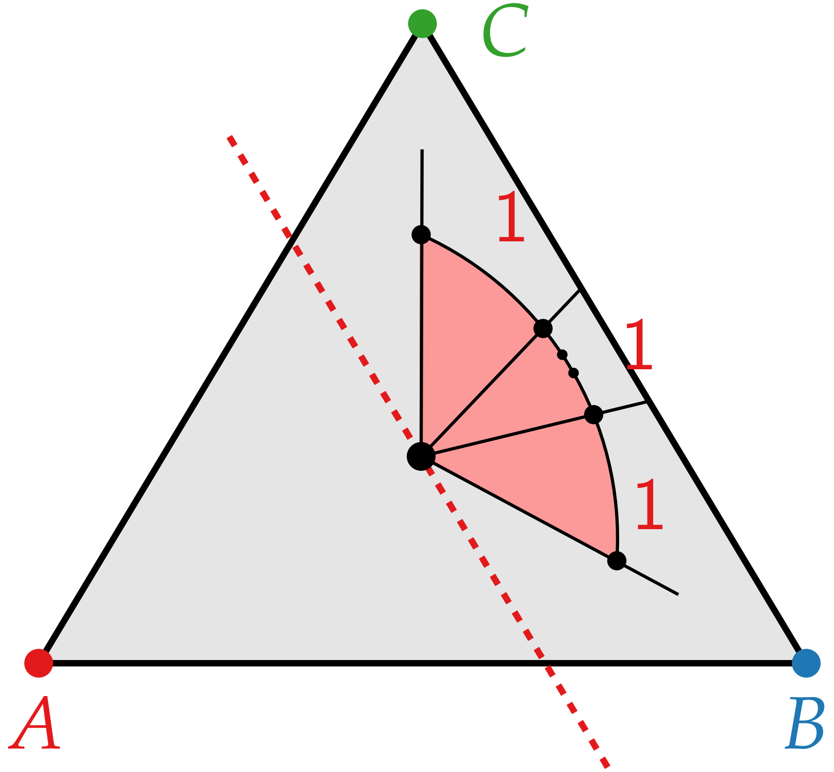

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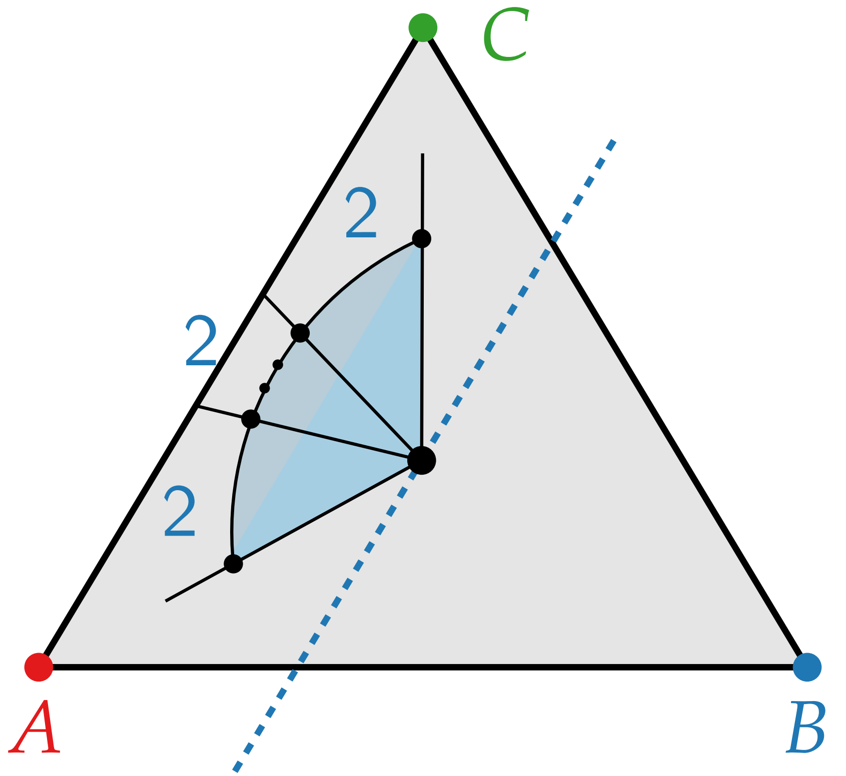

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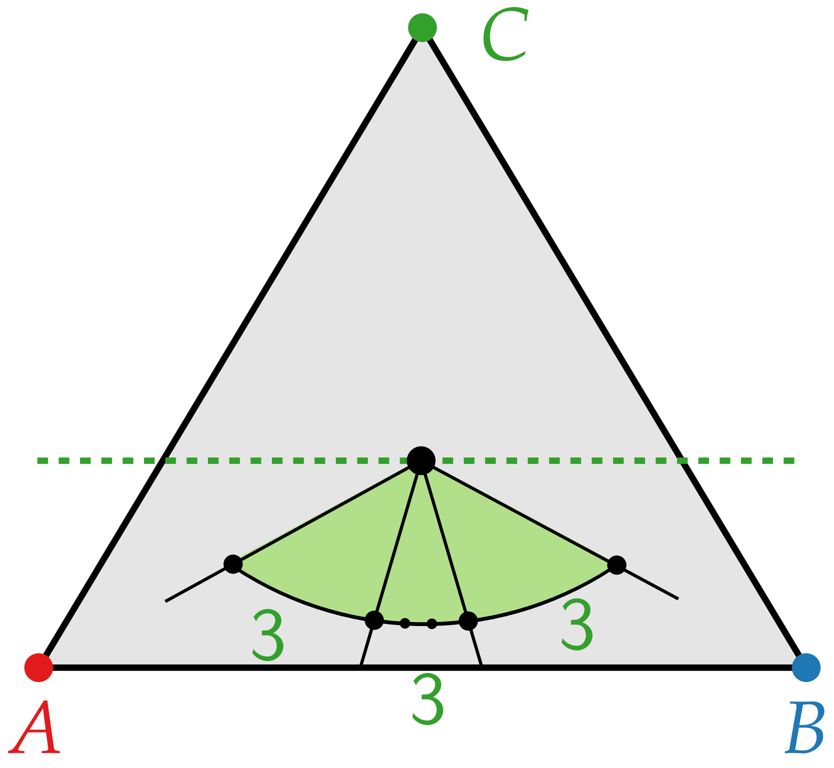
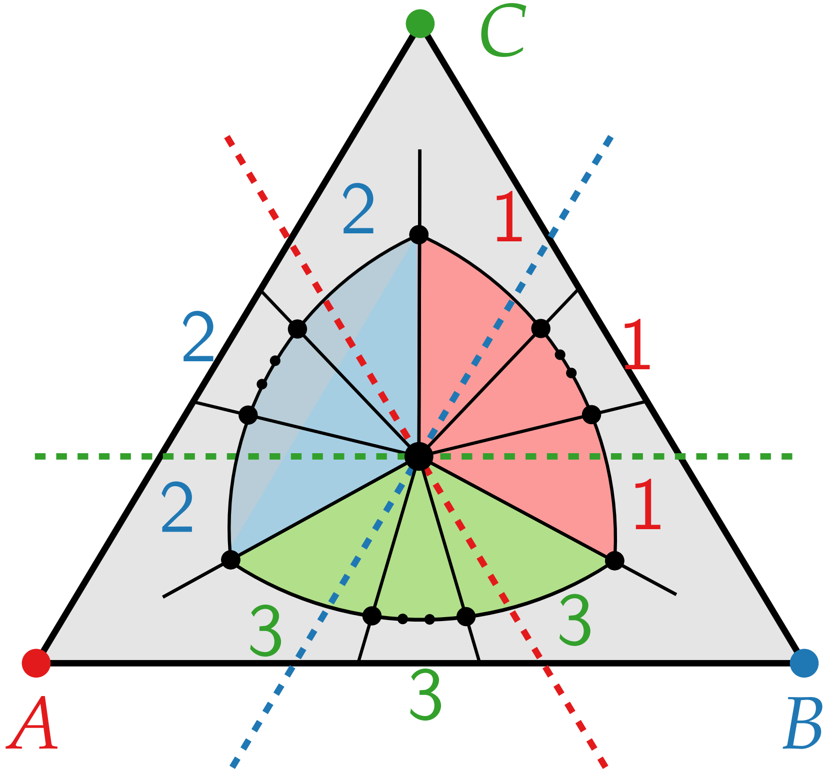


Angle labeling

Observation 2.
Let $v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a triangulated plane graph $G = (V, E)$.
Around a vertex:

- all angles with label i are consecutive
- all three angles appear



Schnyder labeling

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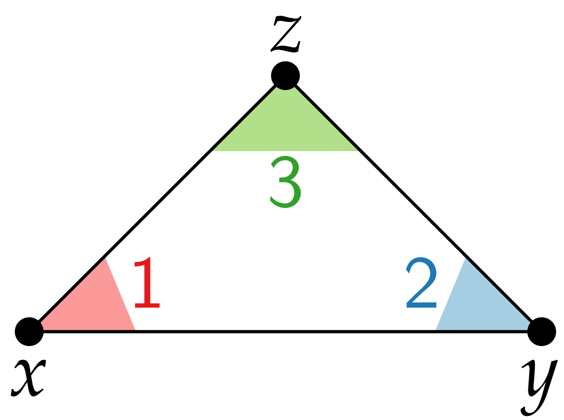

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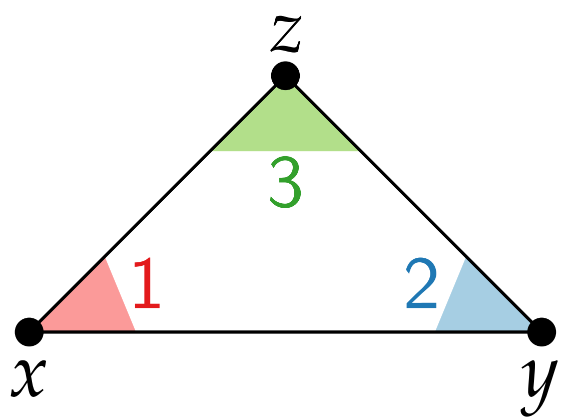
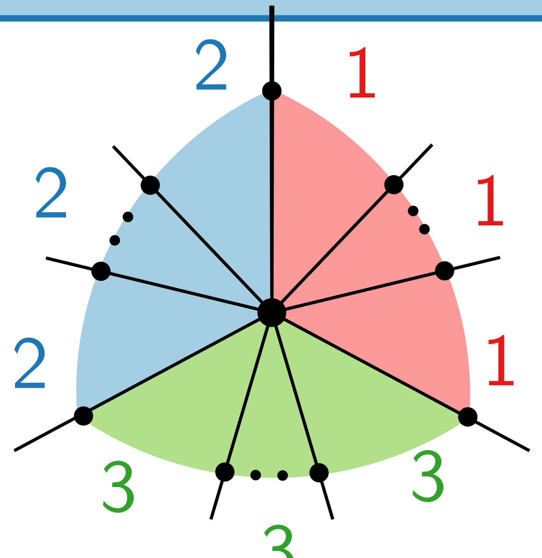
Vertices The ccw order of labels around each vertex consists of a nonempty interval of 1's followed by a nonempty interval of 2's followed by a nonempty interval of 3's.





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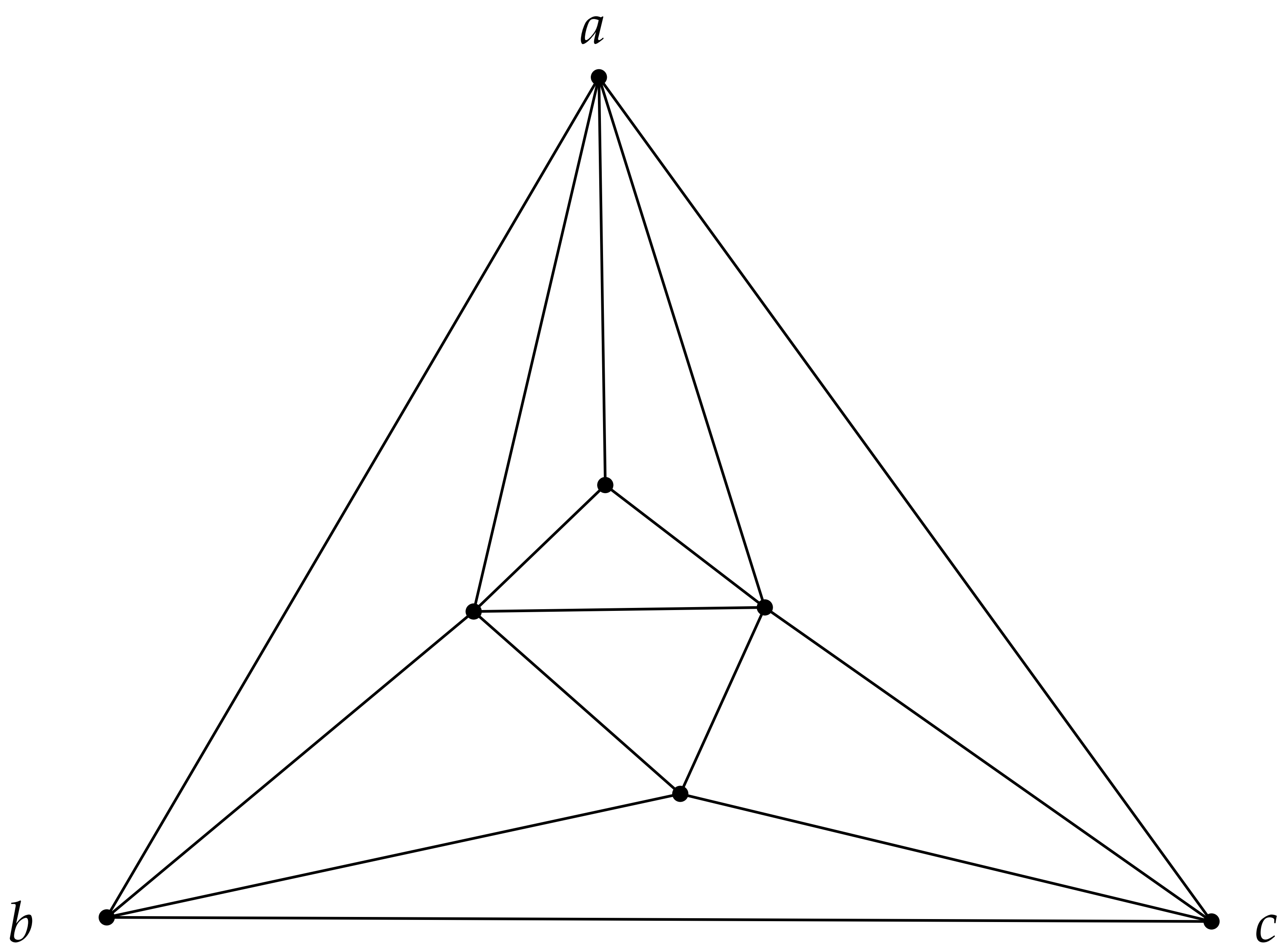

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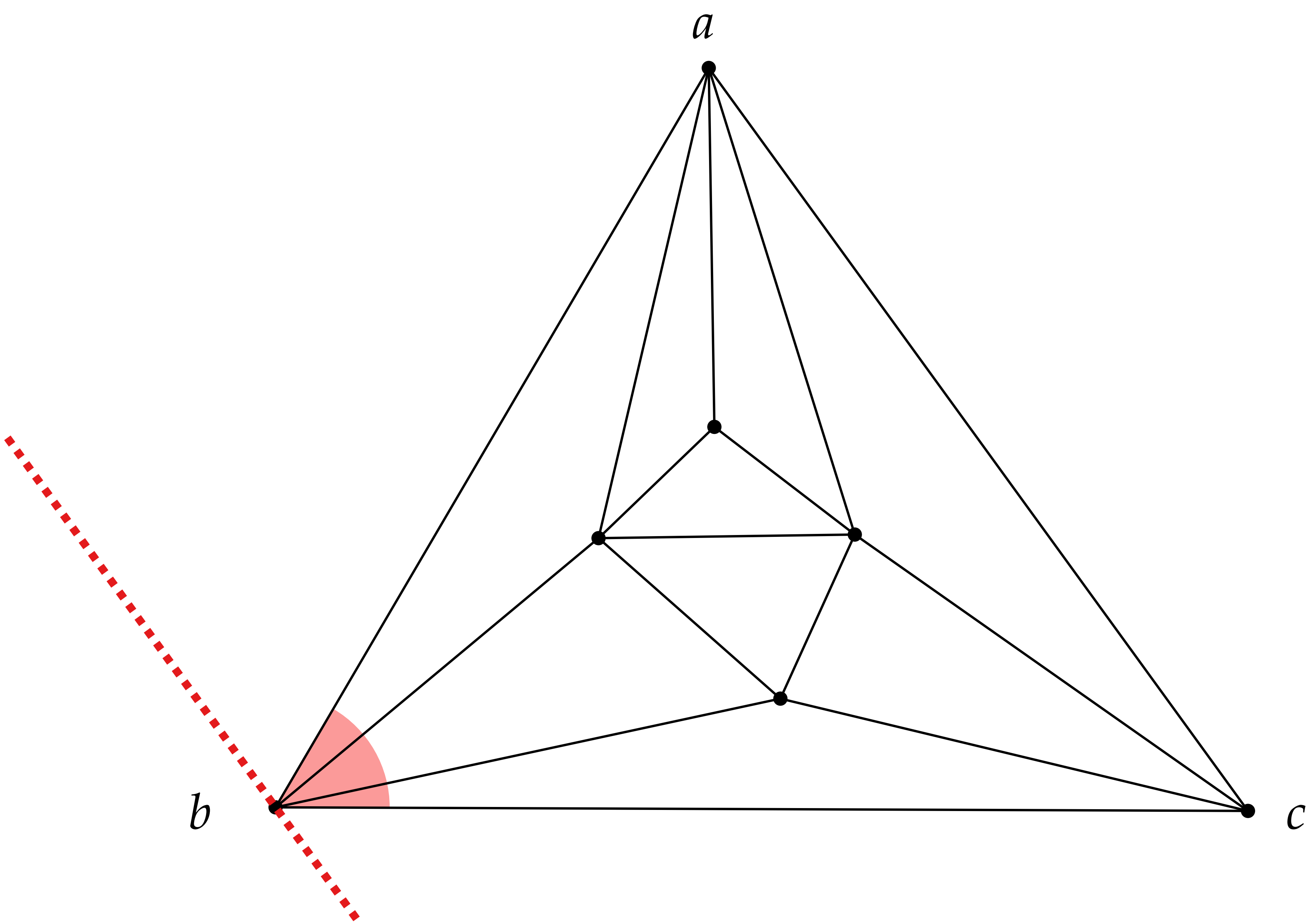

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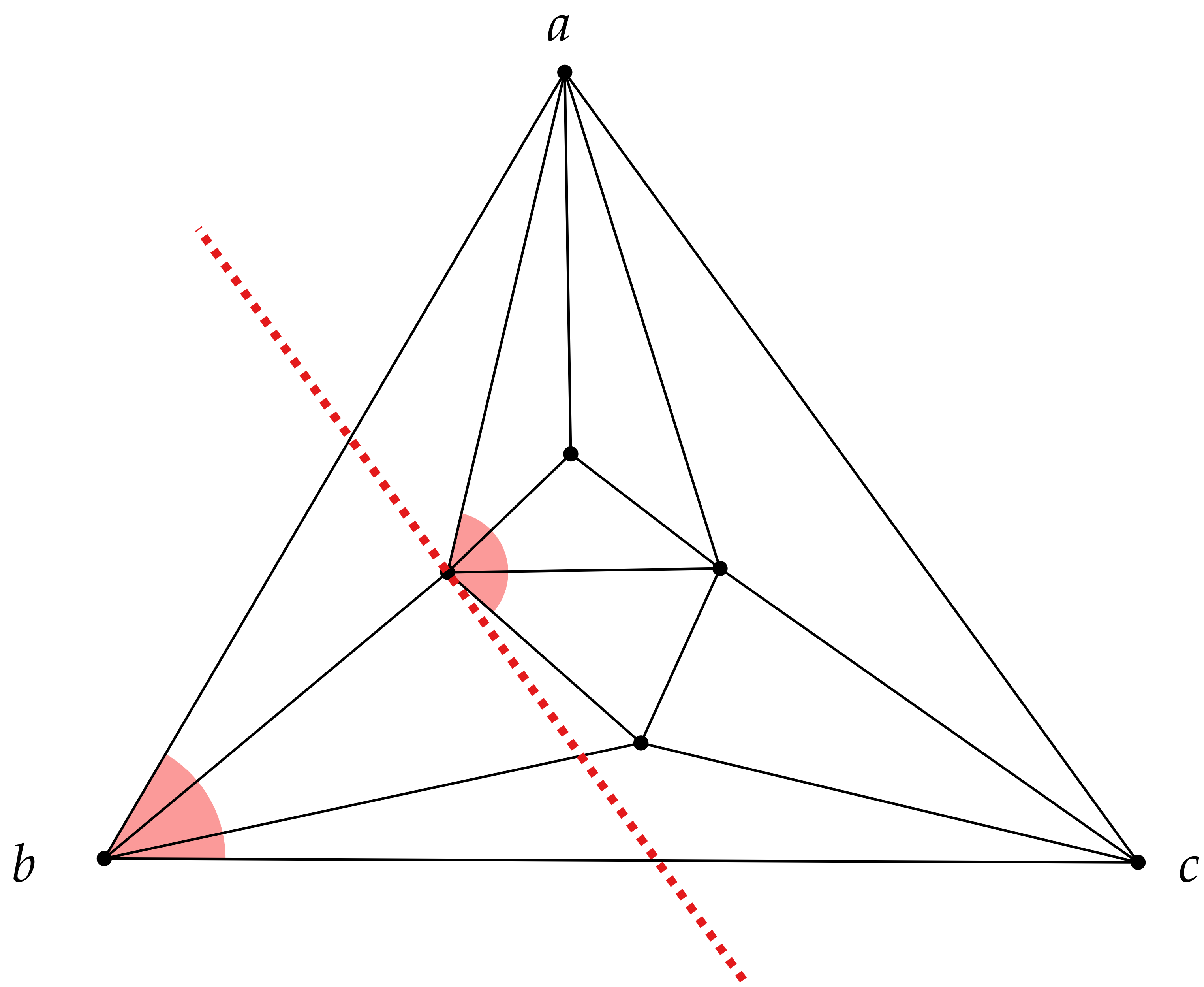

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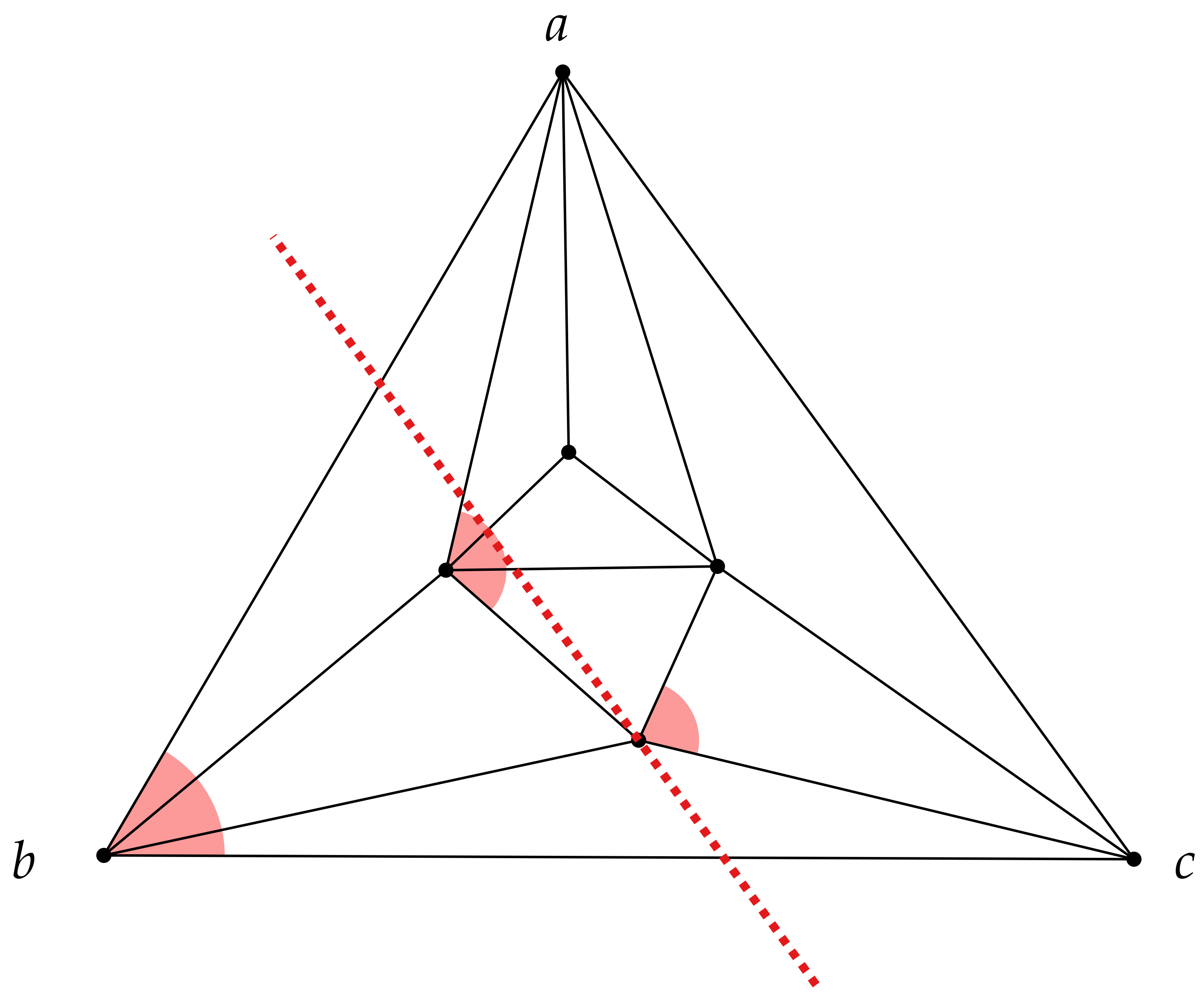

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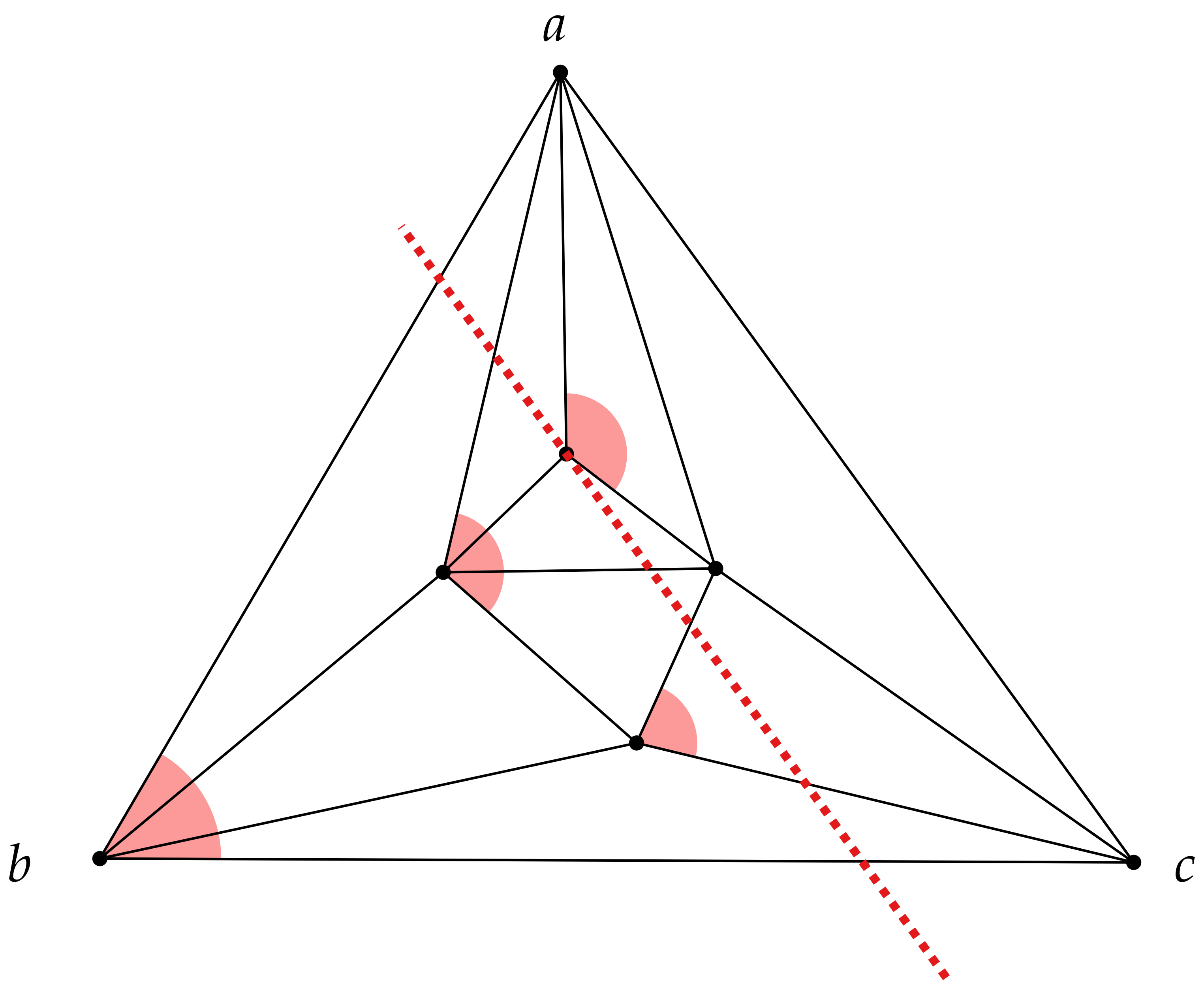

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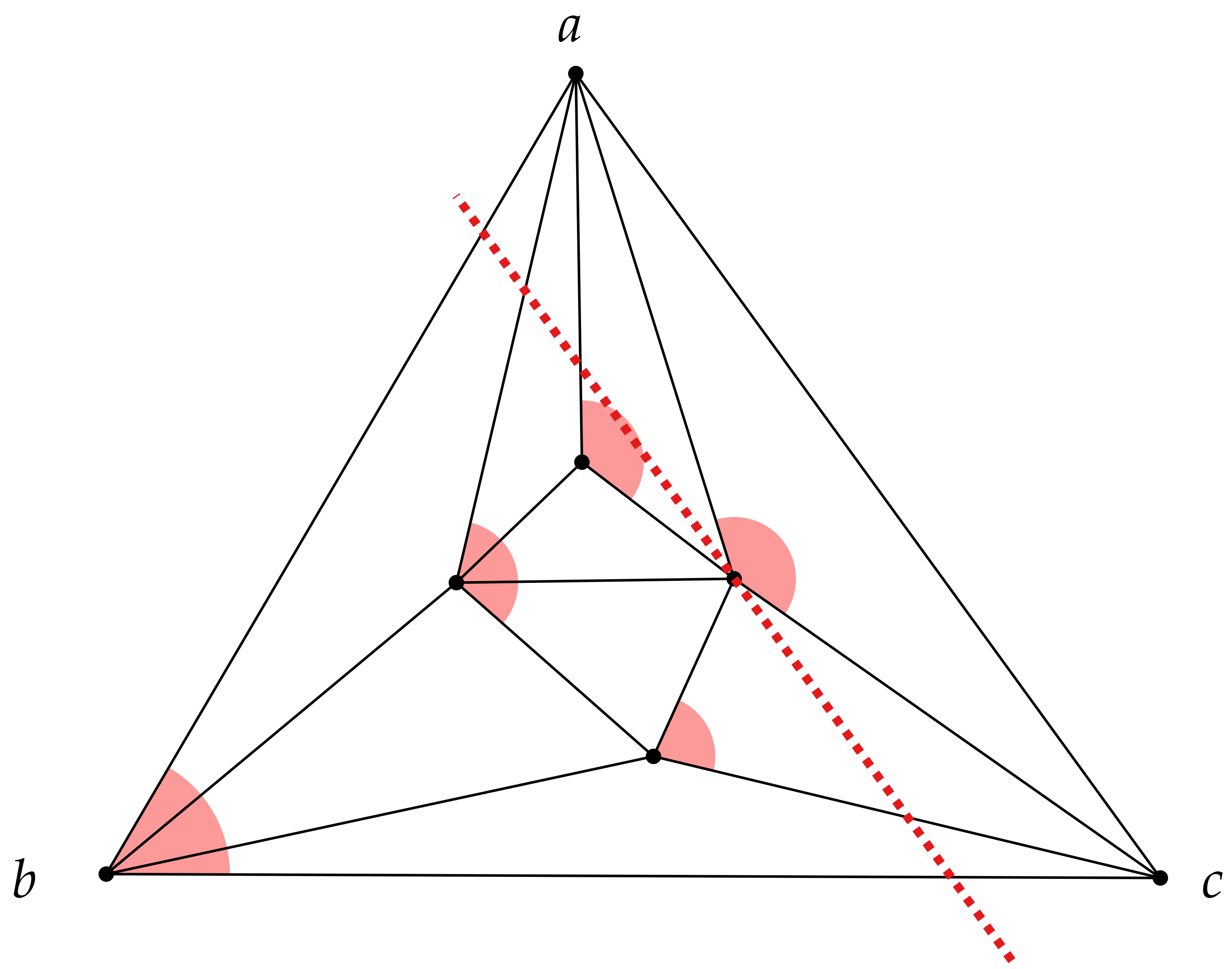

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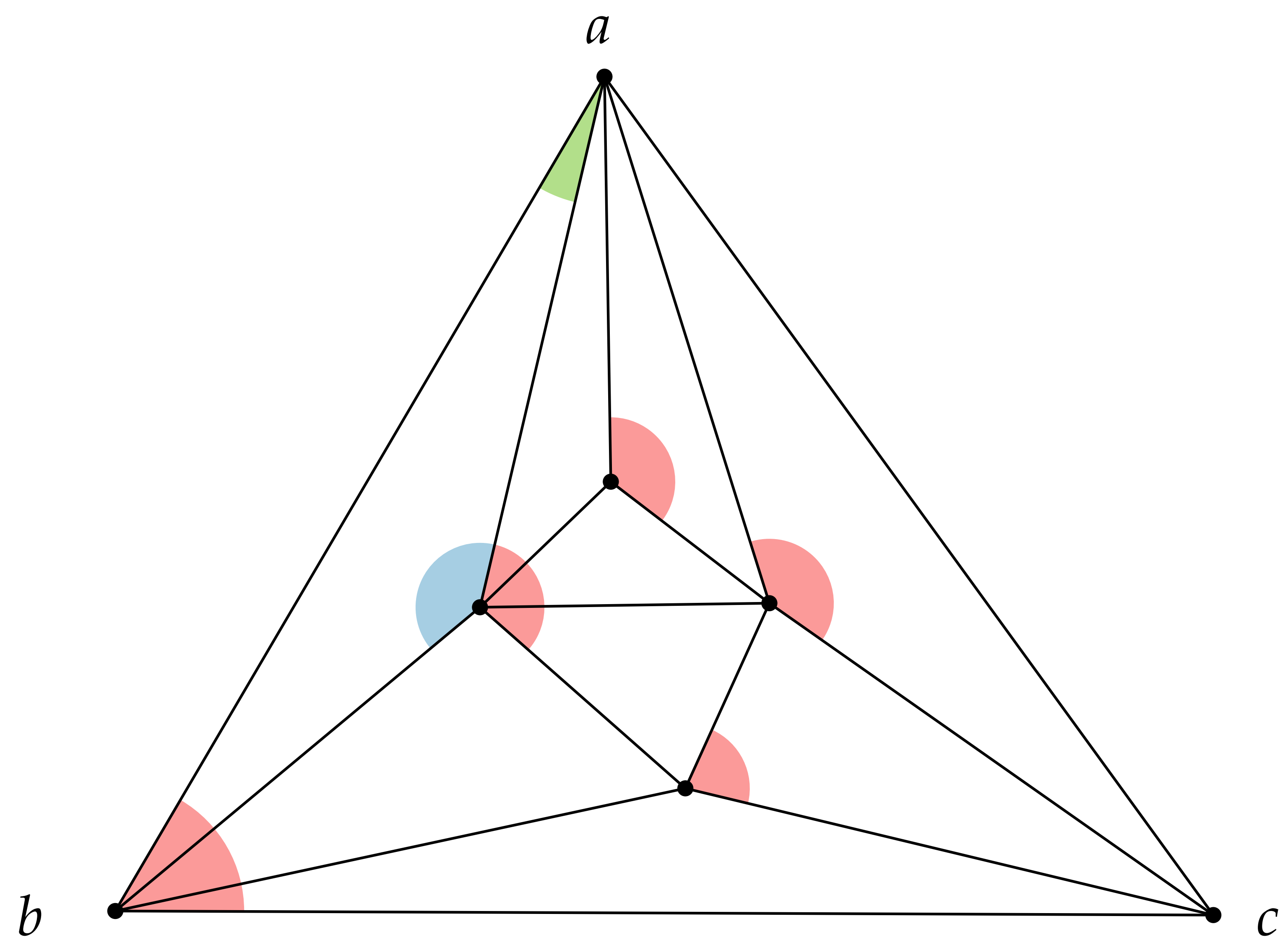

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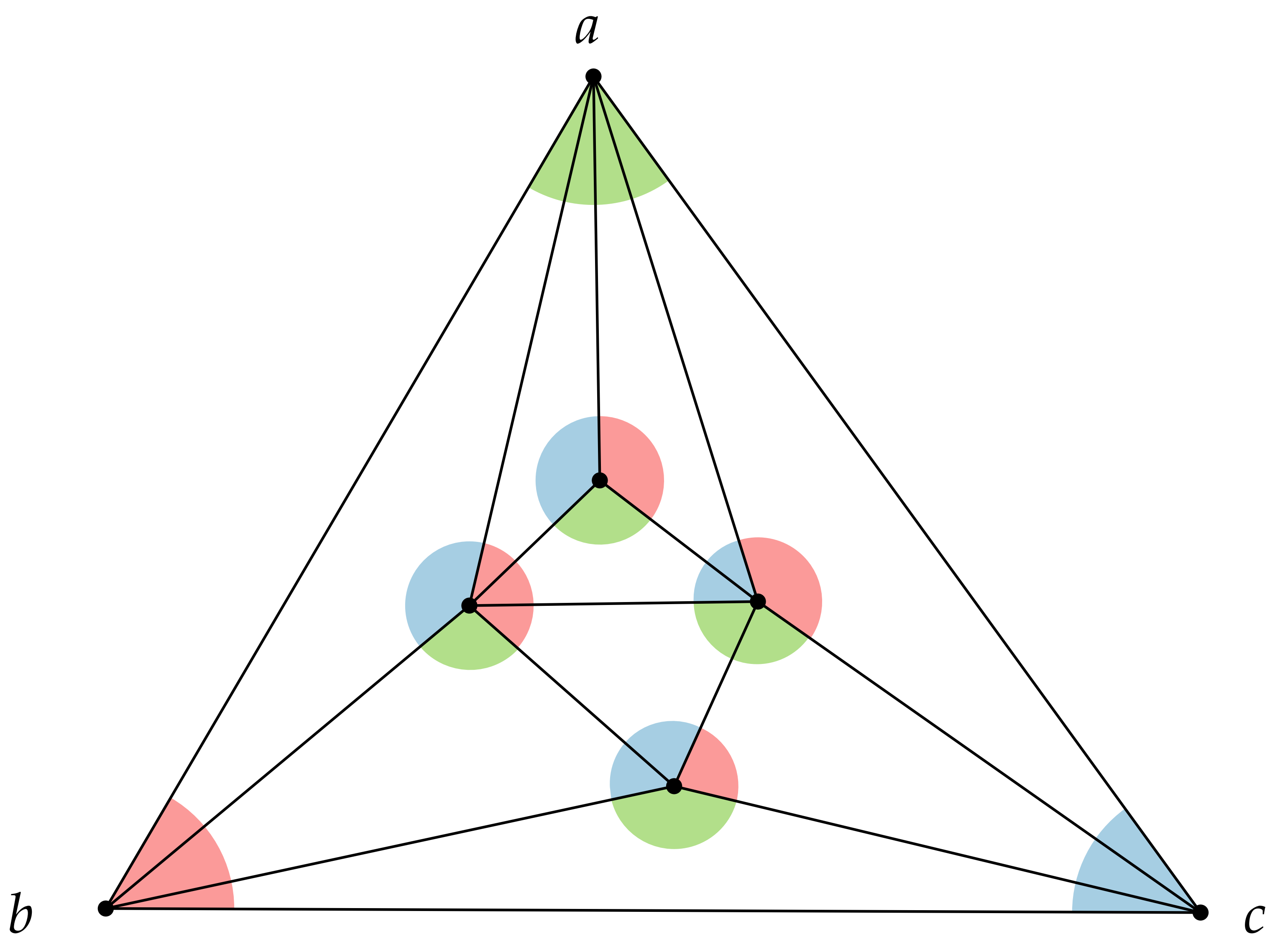

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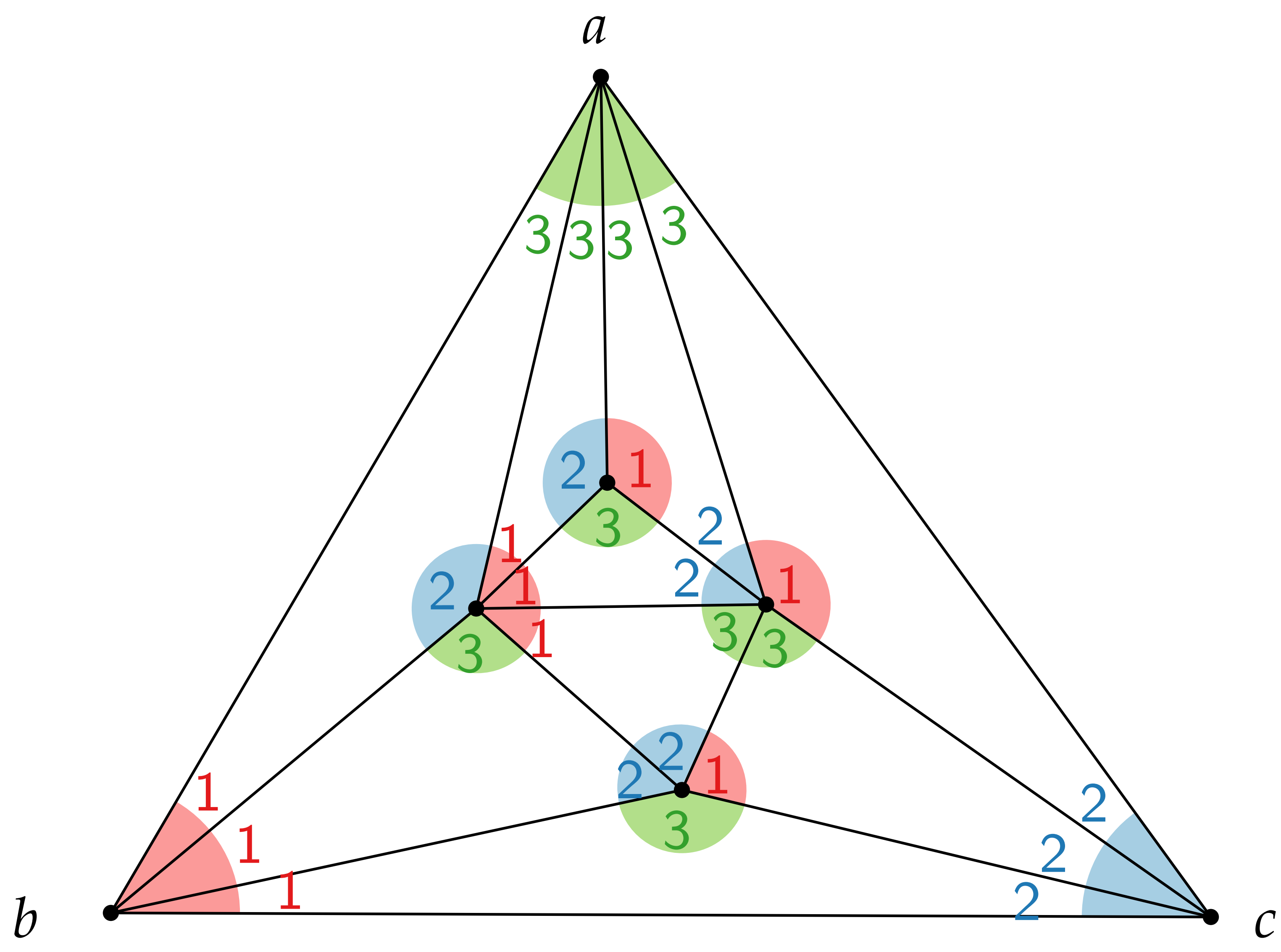

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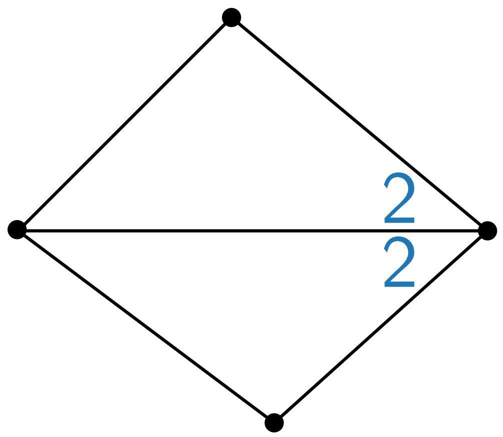

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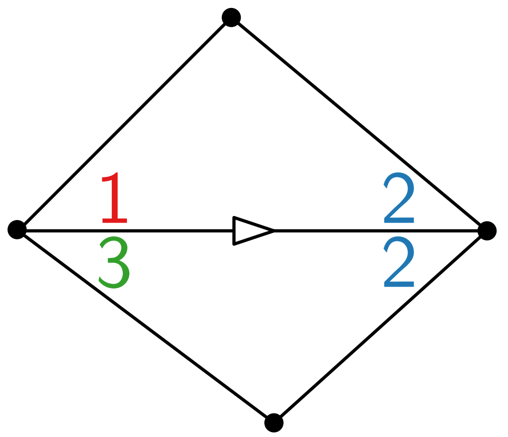

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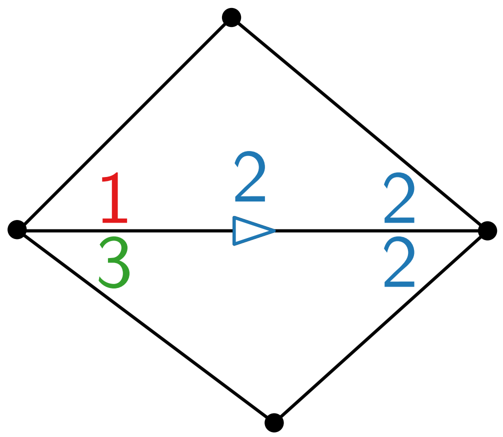

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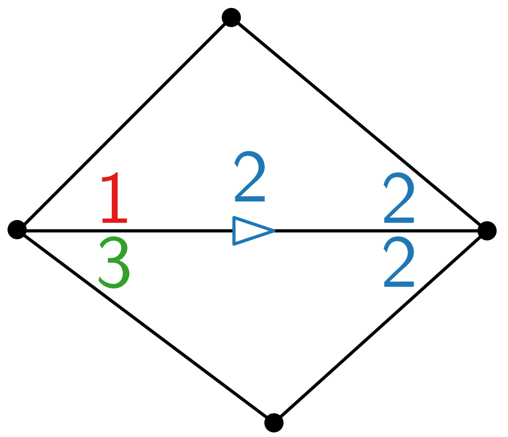

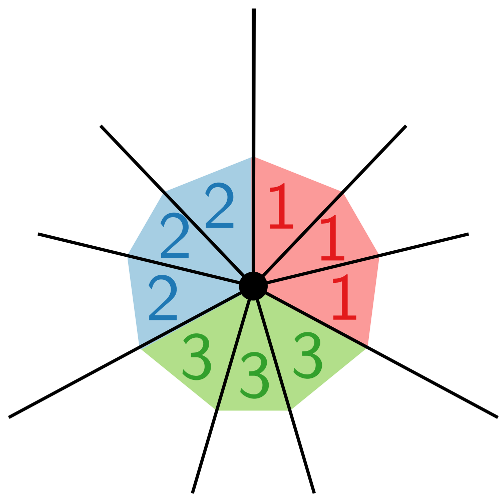

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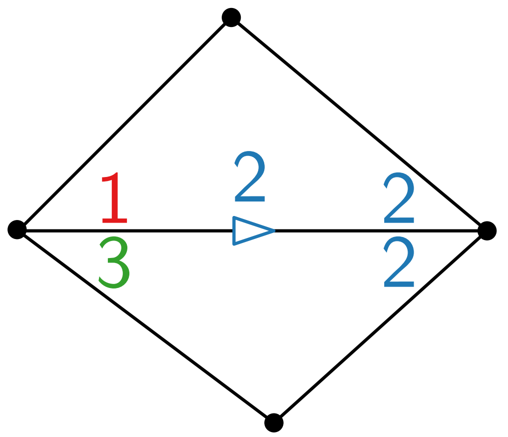

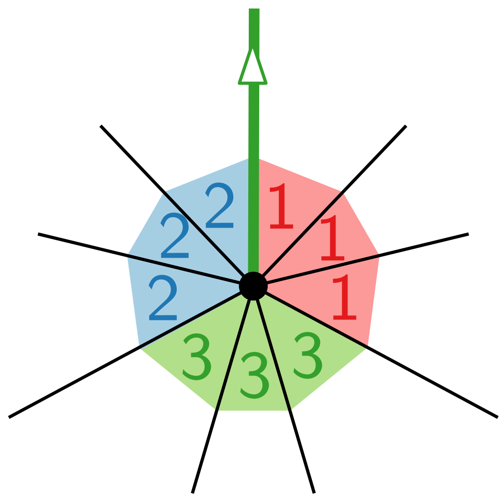

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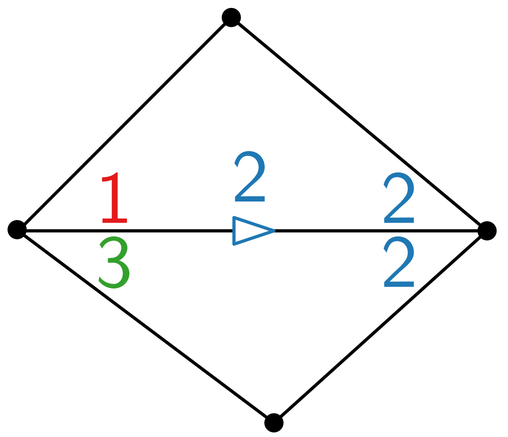

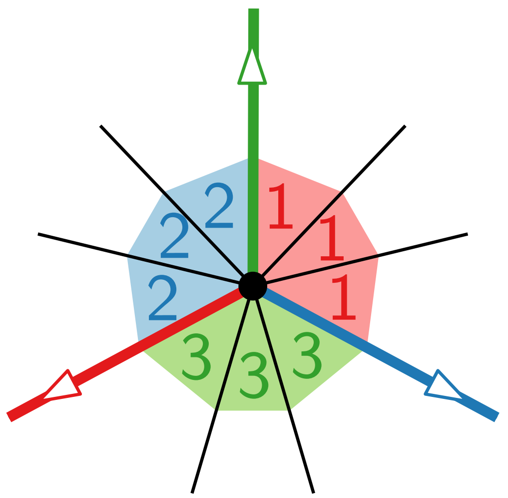

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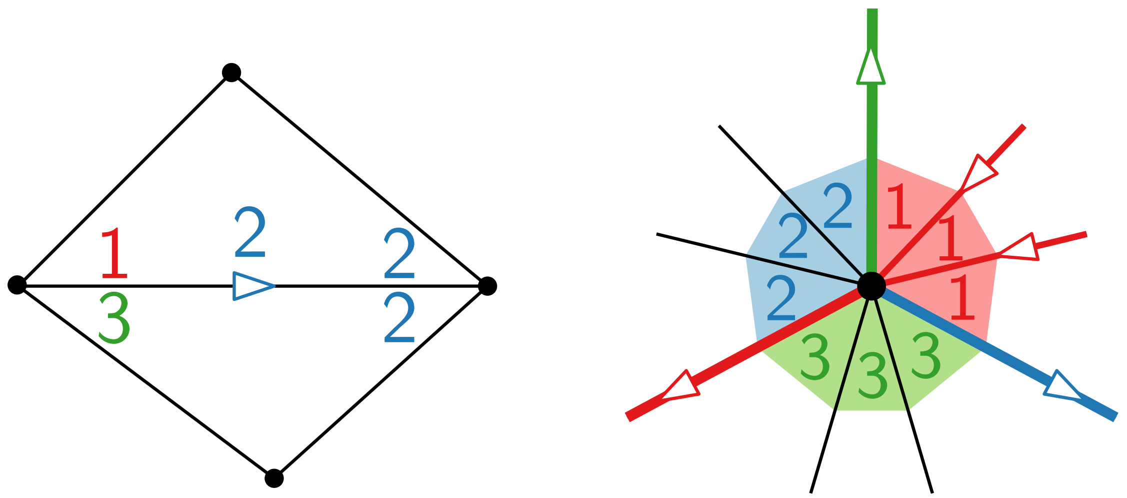

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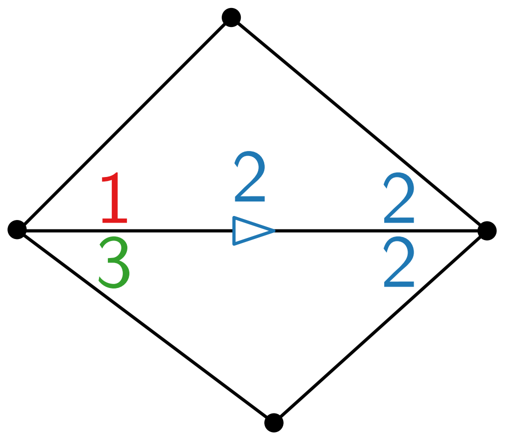

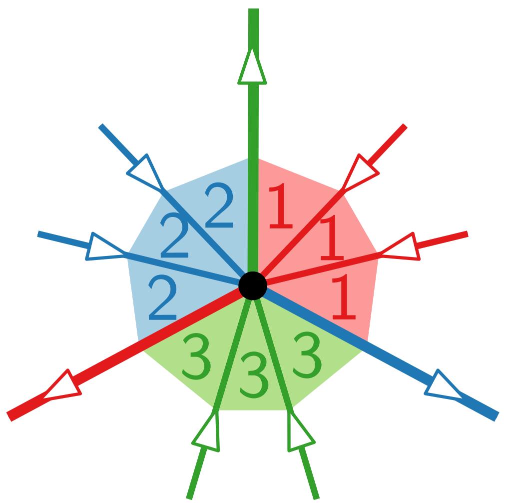

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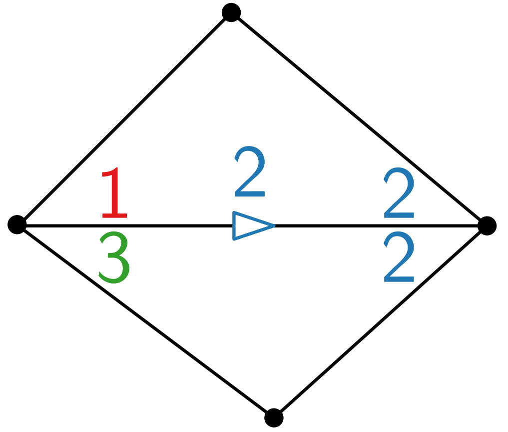

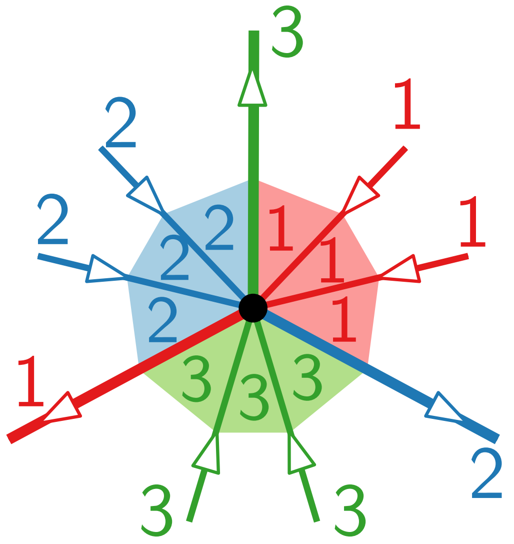

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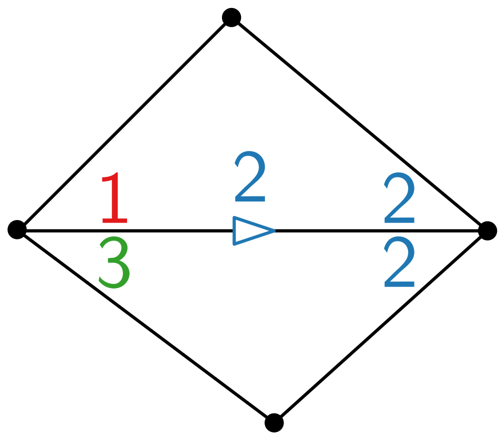

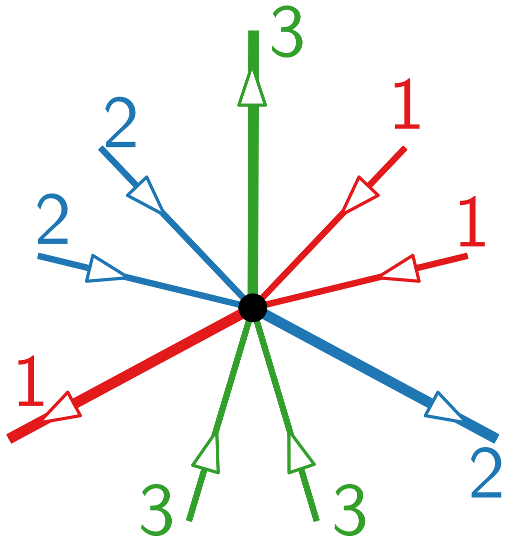

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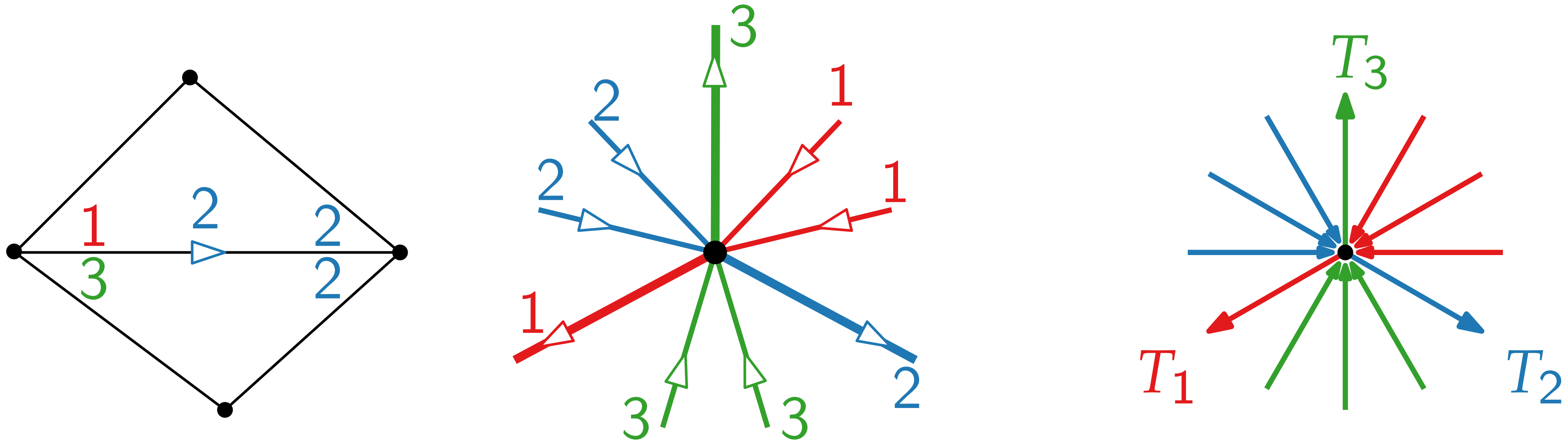

Definition.
A **Schnyder forest** or **realiser** of a triangulated plane graph $G = (V, E)$ is a partition of the inner edges of E into three sets of oriented edges T_1, T_2, T_3 such that for each inner vertex $v \in V$ holds:

- v has one outgoing edge in each of T_1, T_2, and T_3.
- The ccw order of edges around v is: leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3, entering in T_2.

Schnyder realiser – existence

Lemma. [Kampen 1976]
Let G be a triangulated plane graph with vertices a, b, c on the outer face. There exists a **contractible edge** $\{a, x\}$ in G, $x \neq b, c$.

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Edge $\{a, x\}$, where $x \neq b, c$, is a **contractible** edge in G, if

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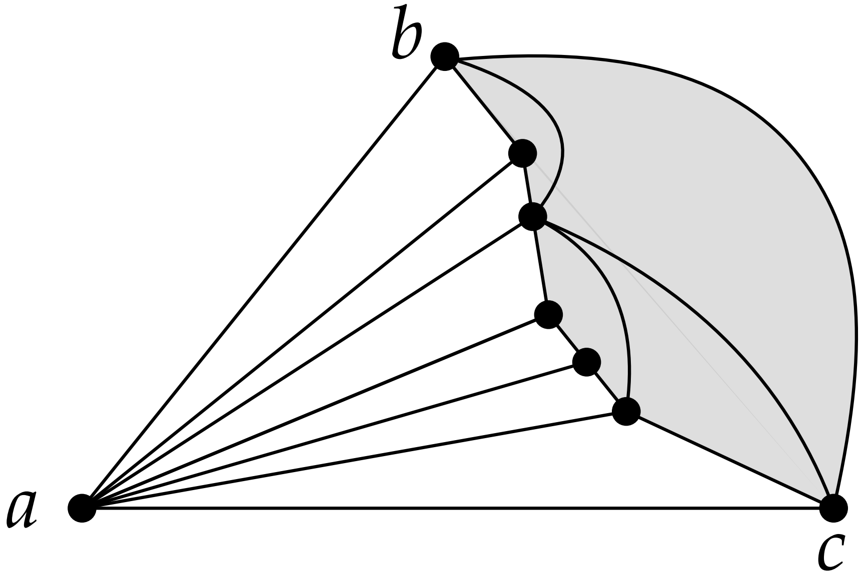

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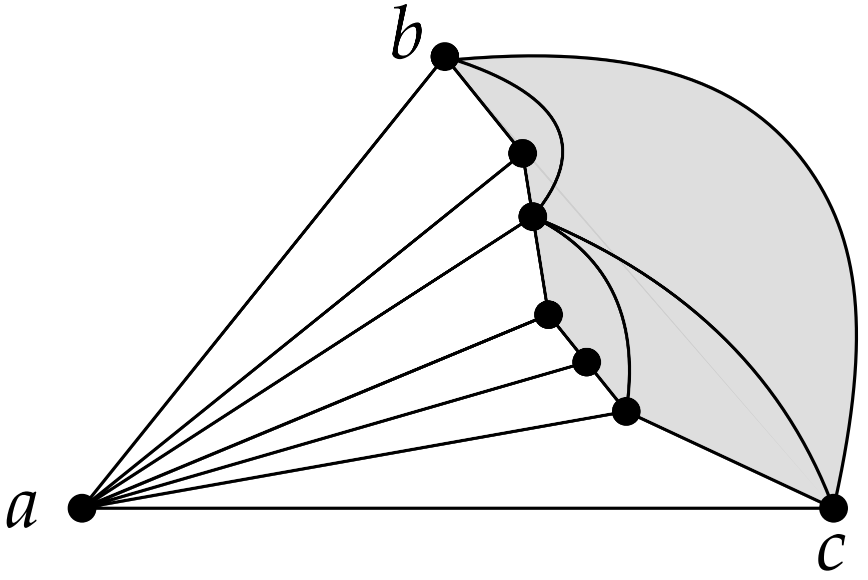

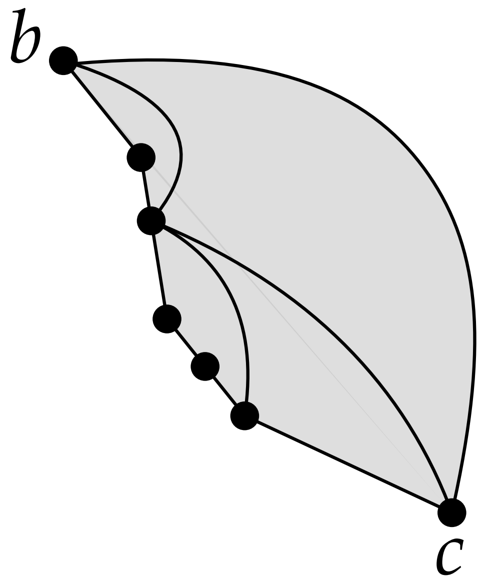

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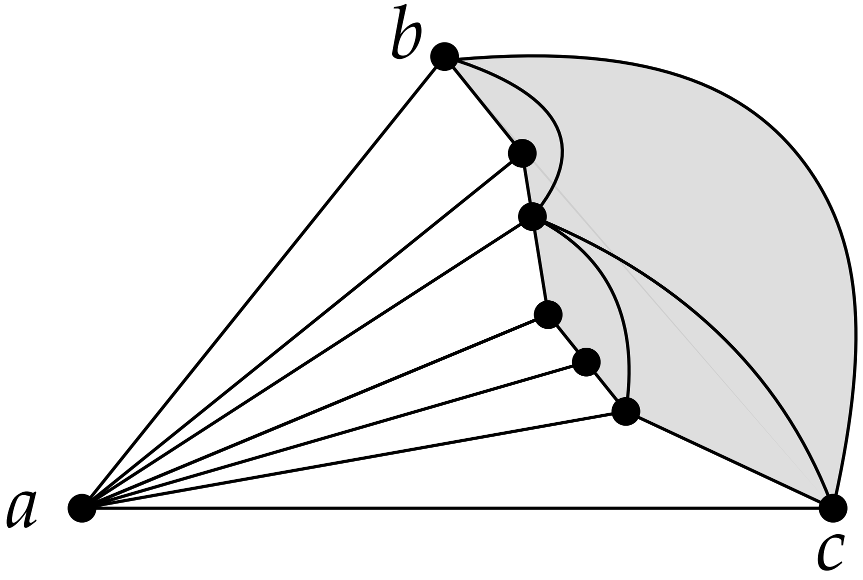

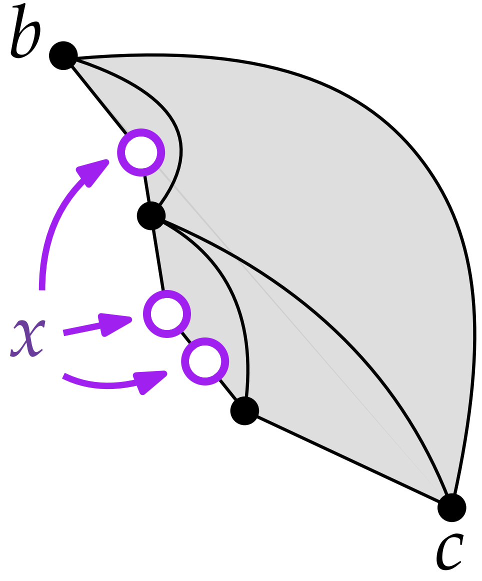

- Neighbors of a induce an outerplanar graph
- There exists $x \neq b, c$ with degree 2

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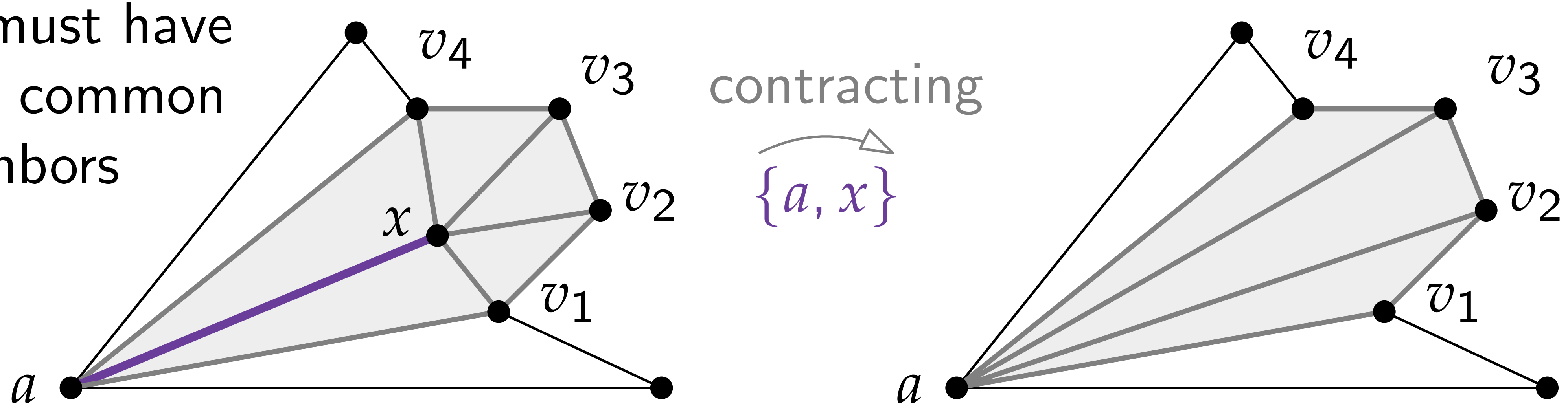

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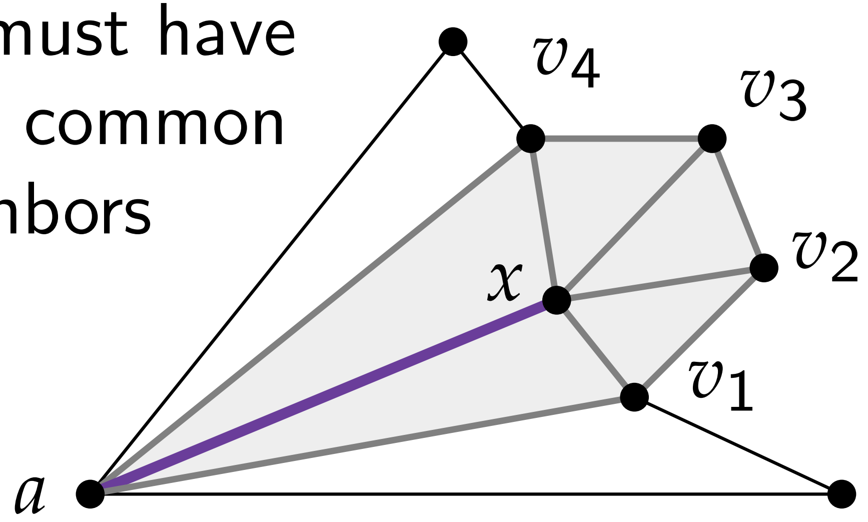
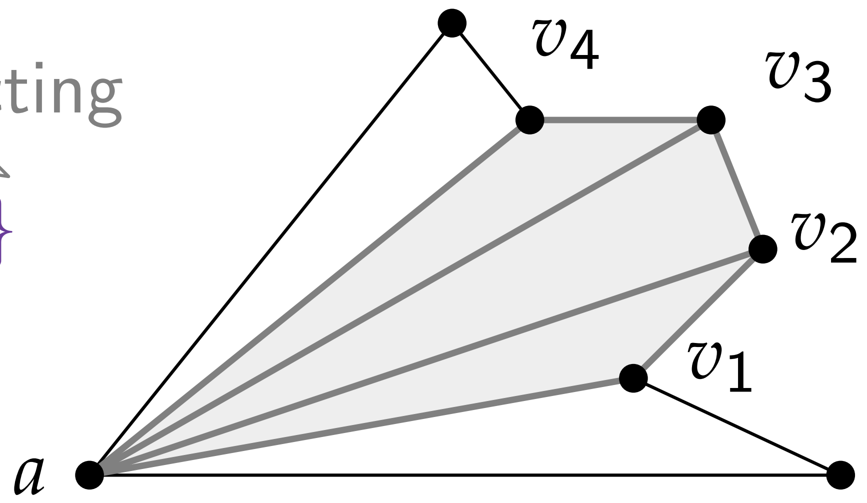


contracting

$\{a, x\}$



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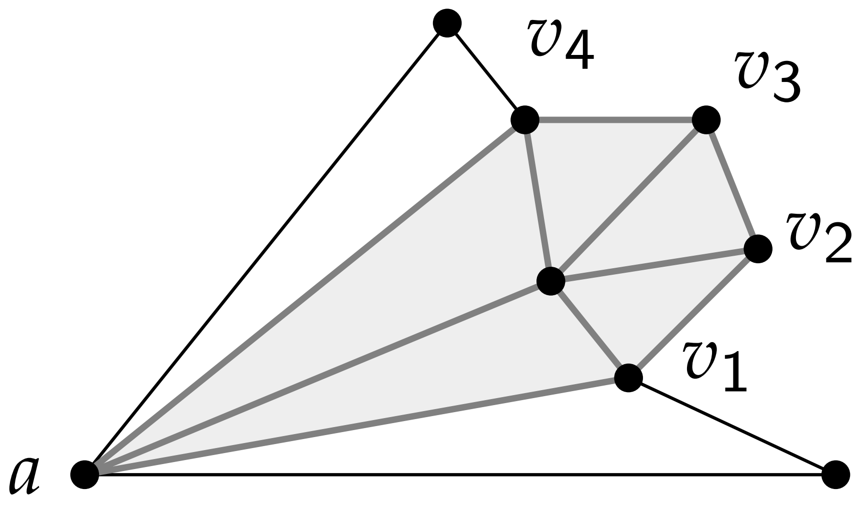

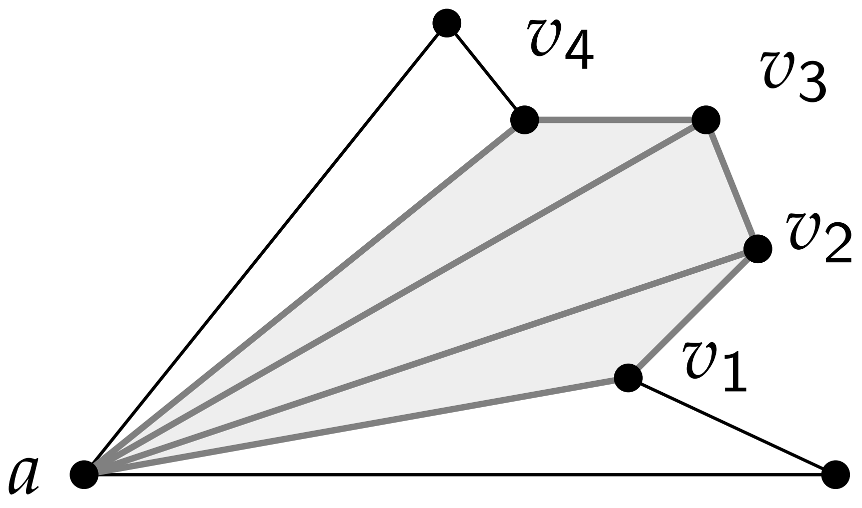

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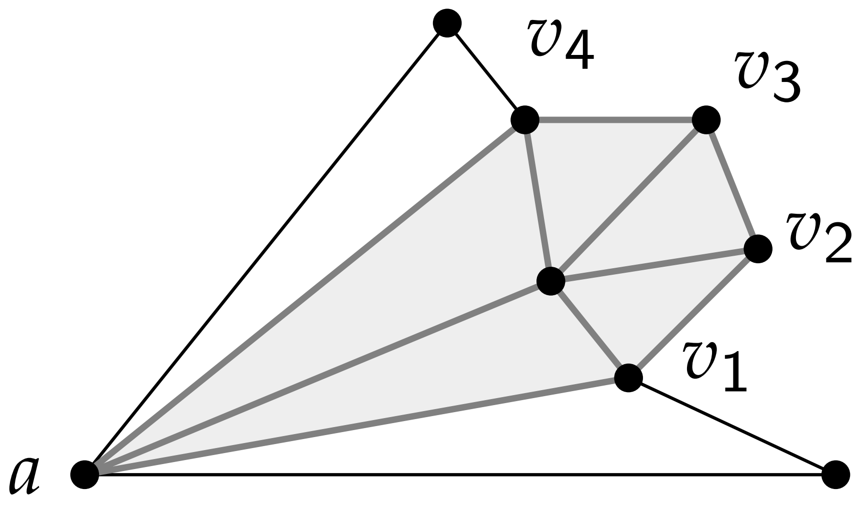

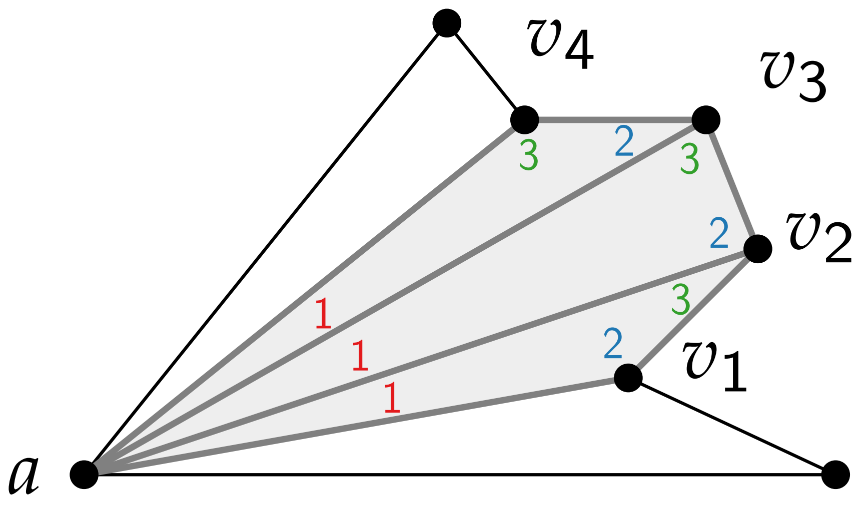

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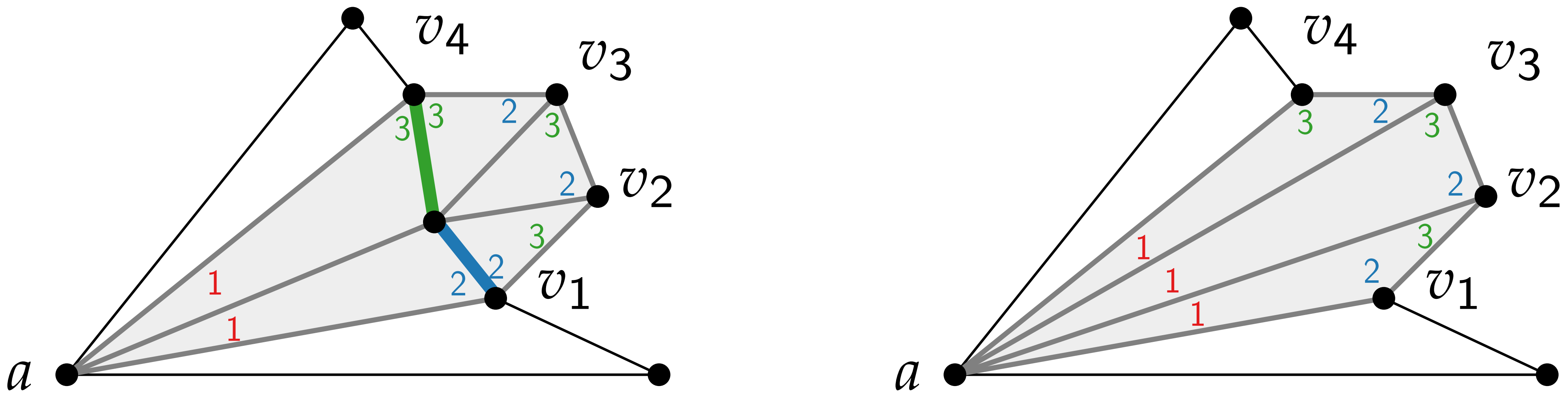

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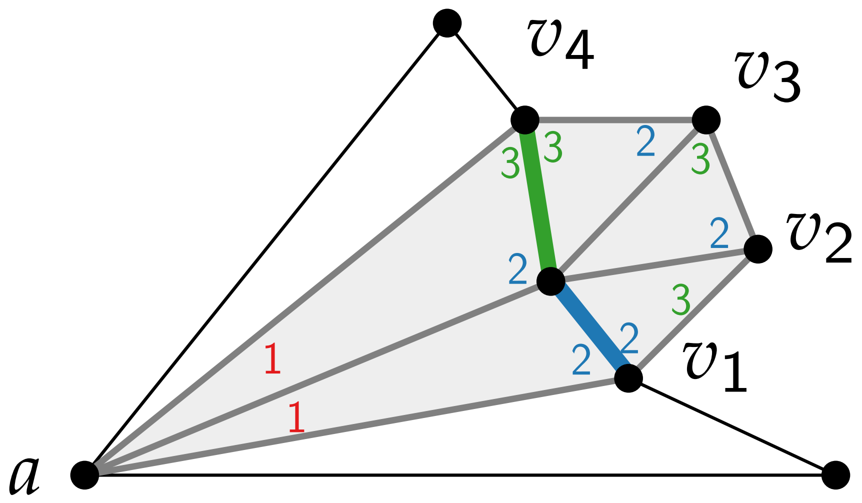

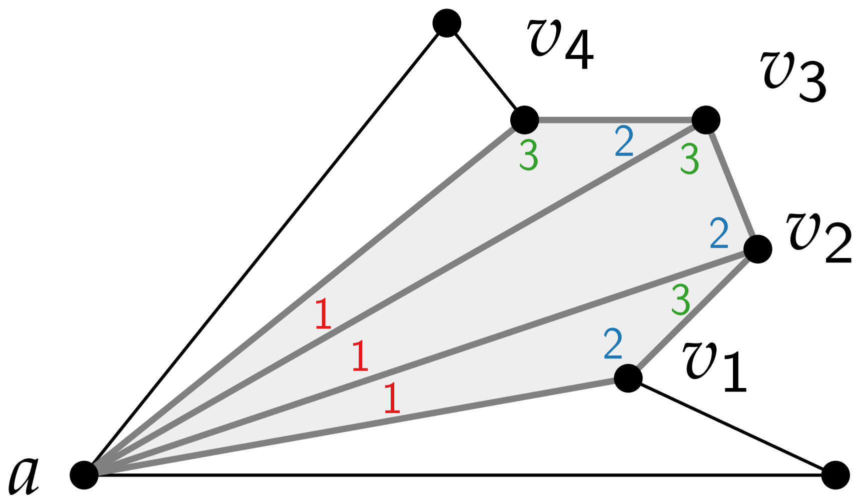

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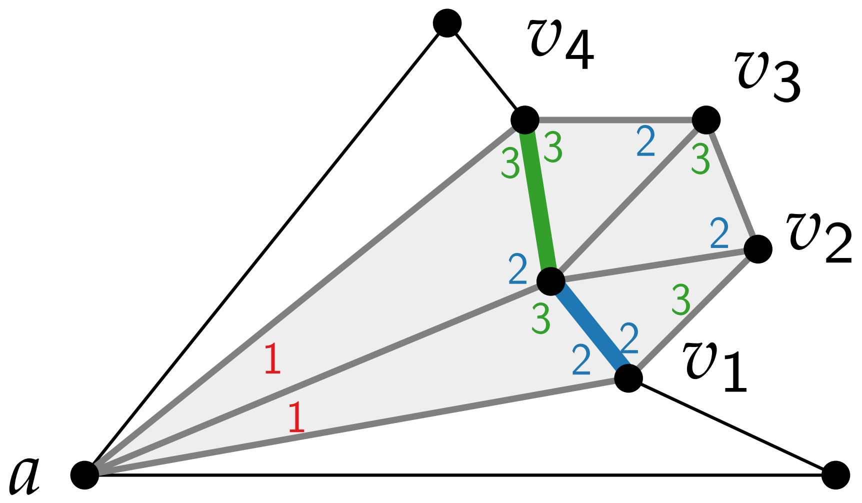

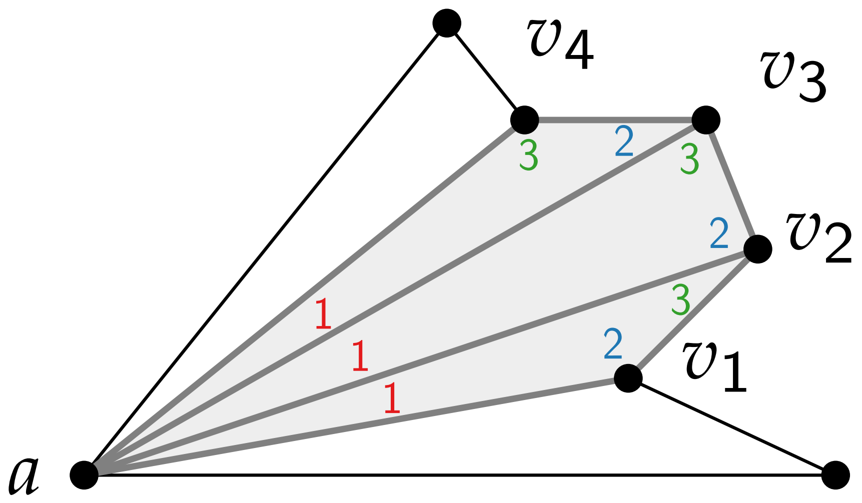

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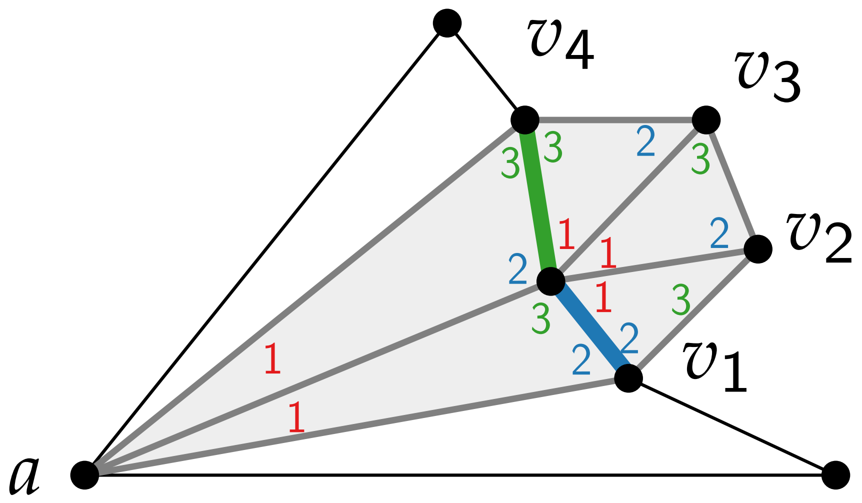

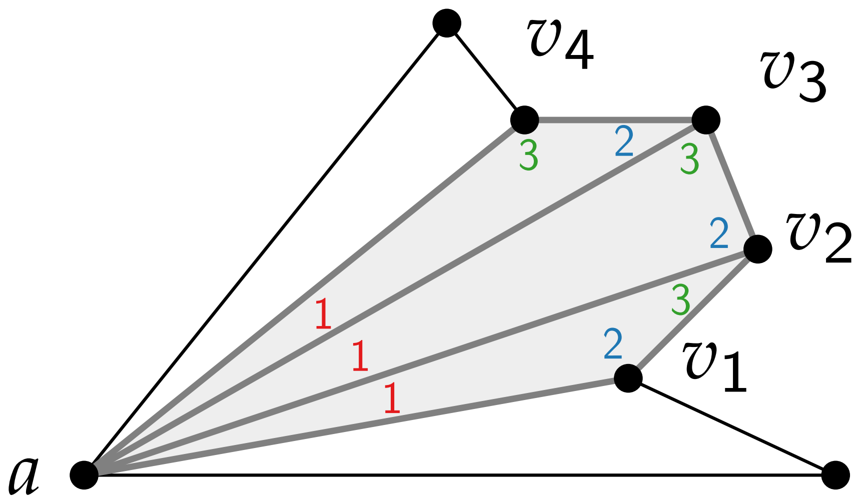

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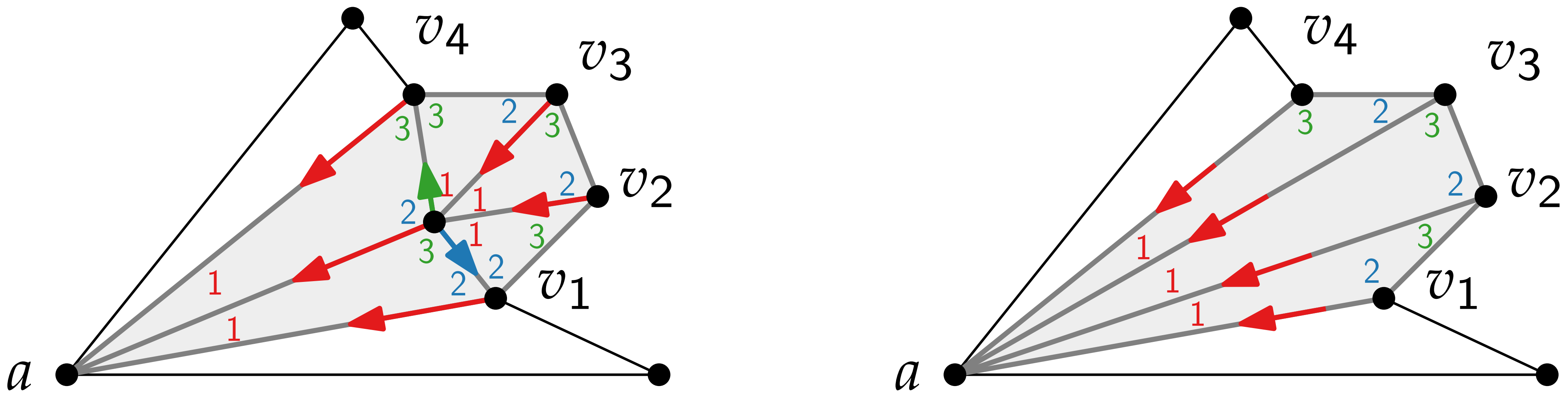

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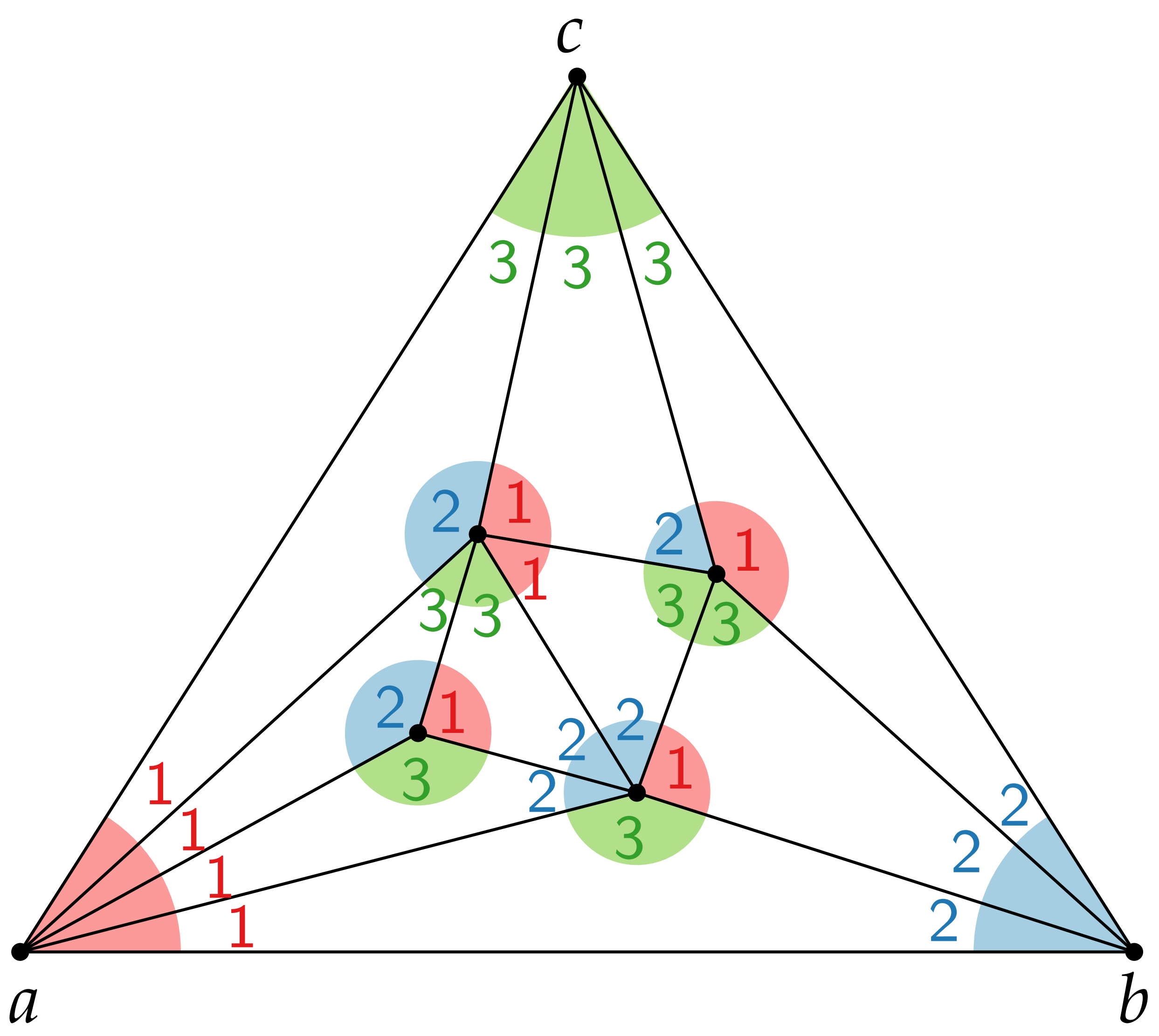
Theorem and previous construction imply:

Corollary.
Every triangulated plane graph has a Schnyder realiser.

Schnyder realiser – properties



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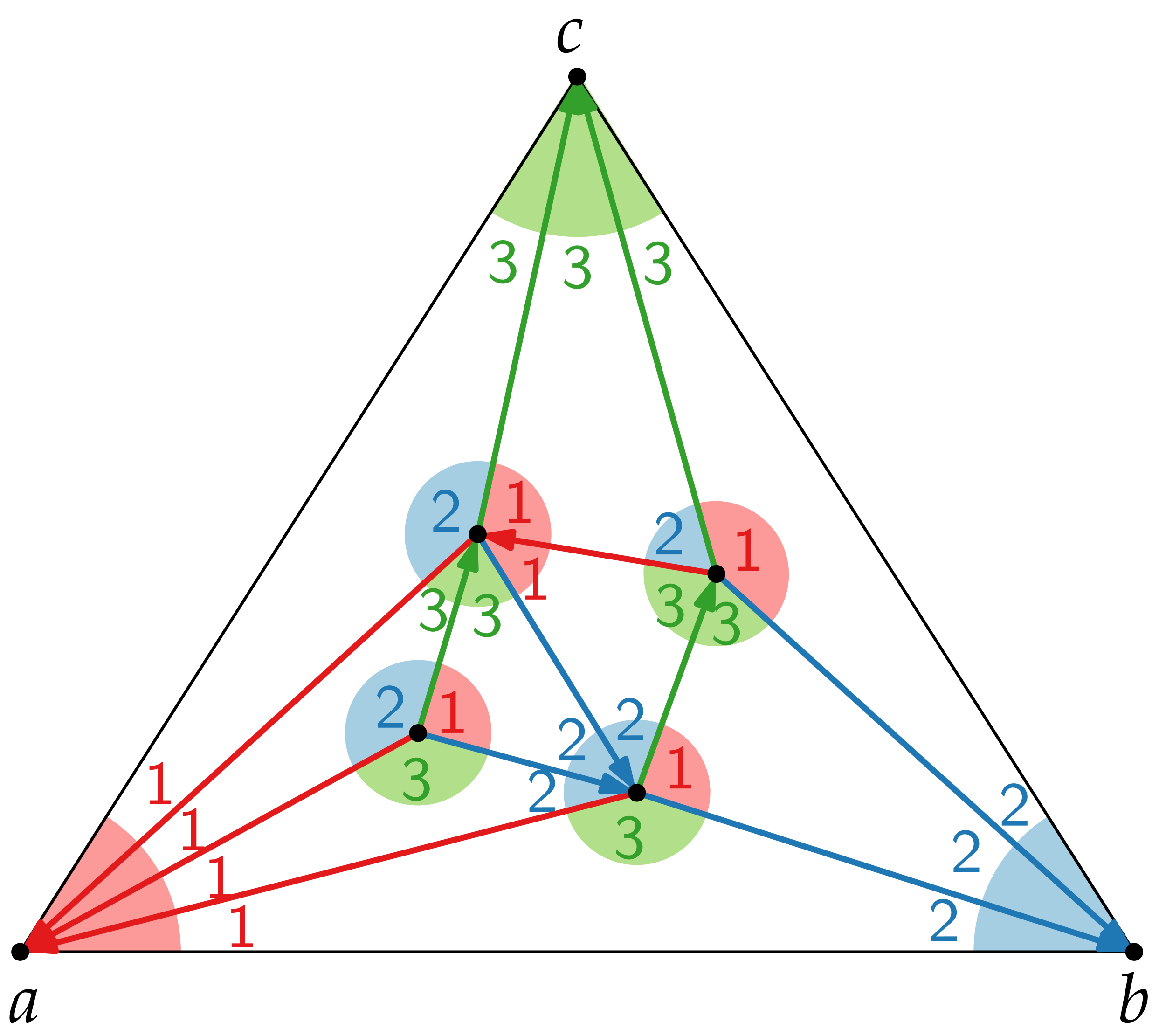

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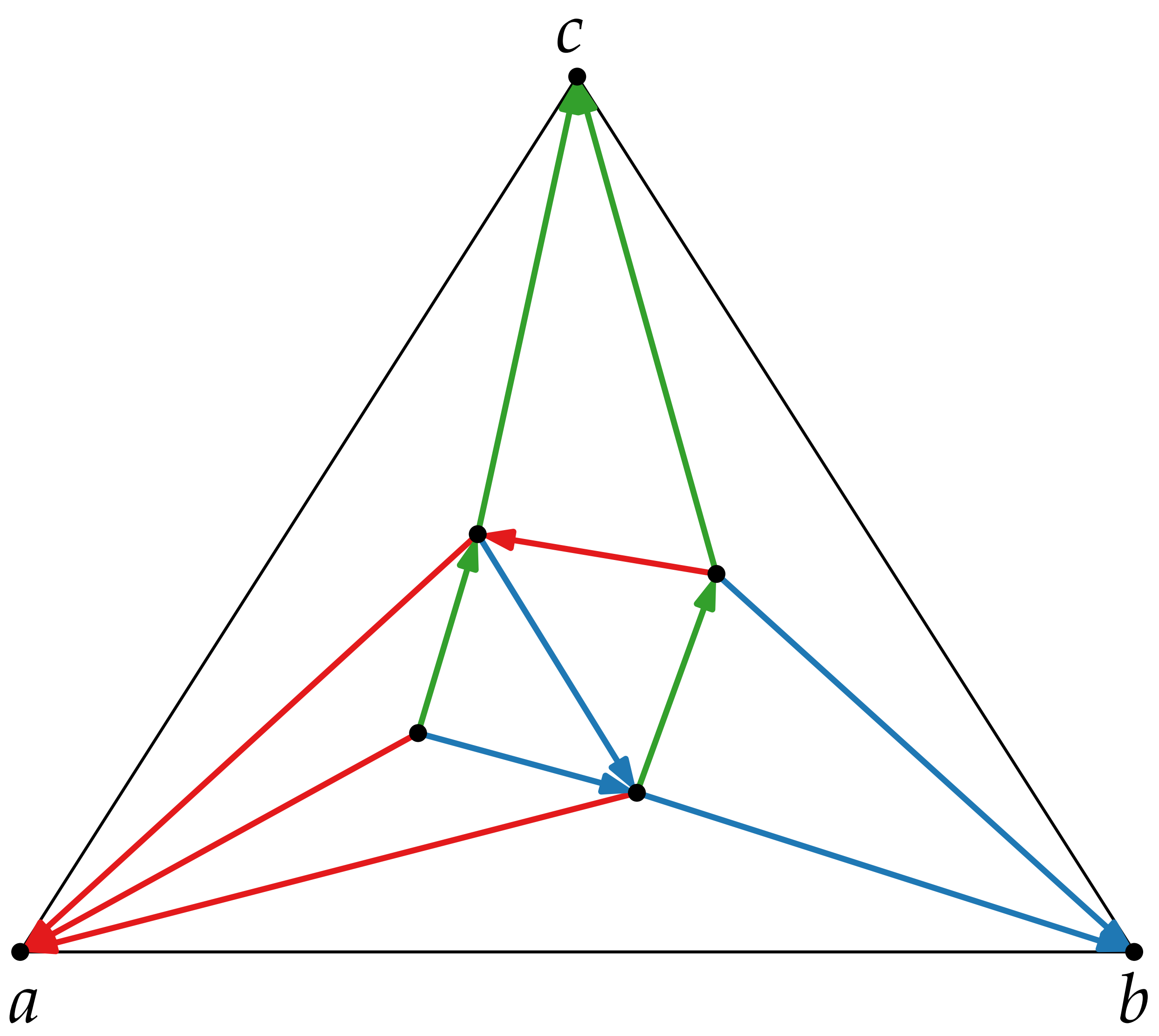
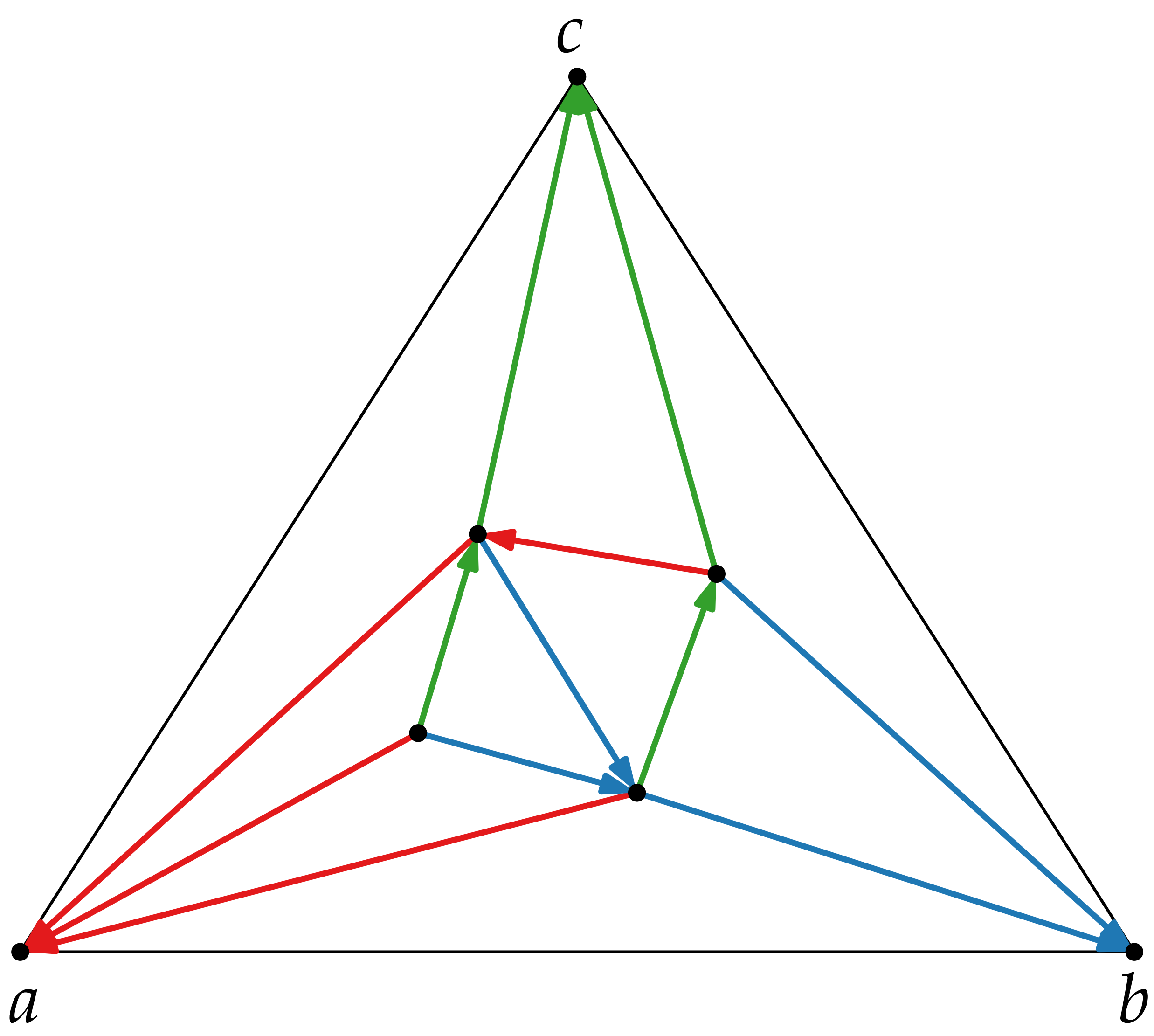


Schnyder realiser – properties



- For each v there exists a directed red, blue, green path from v to a, b, c, respectively.
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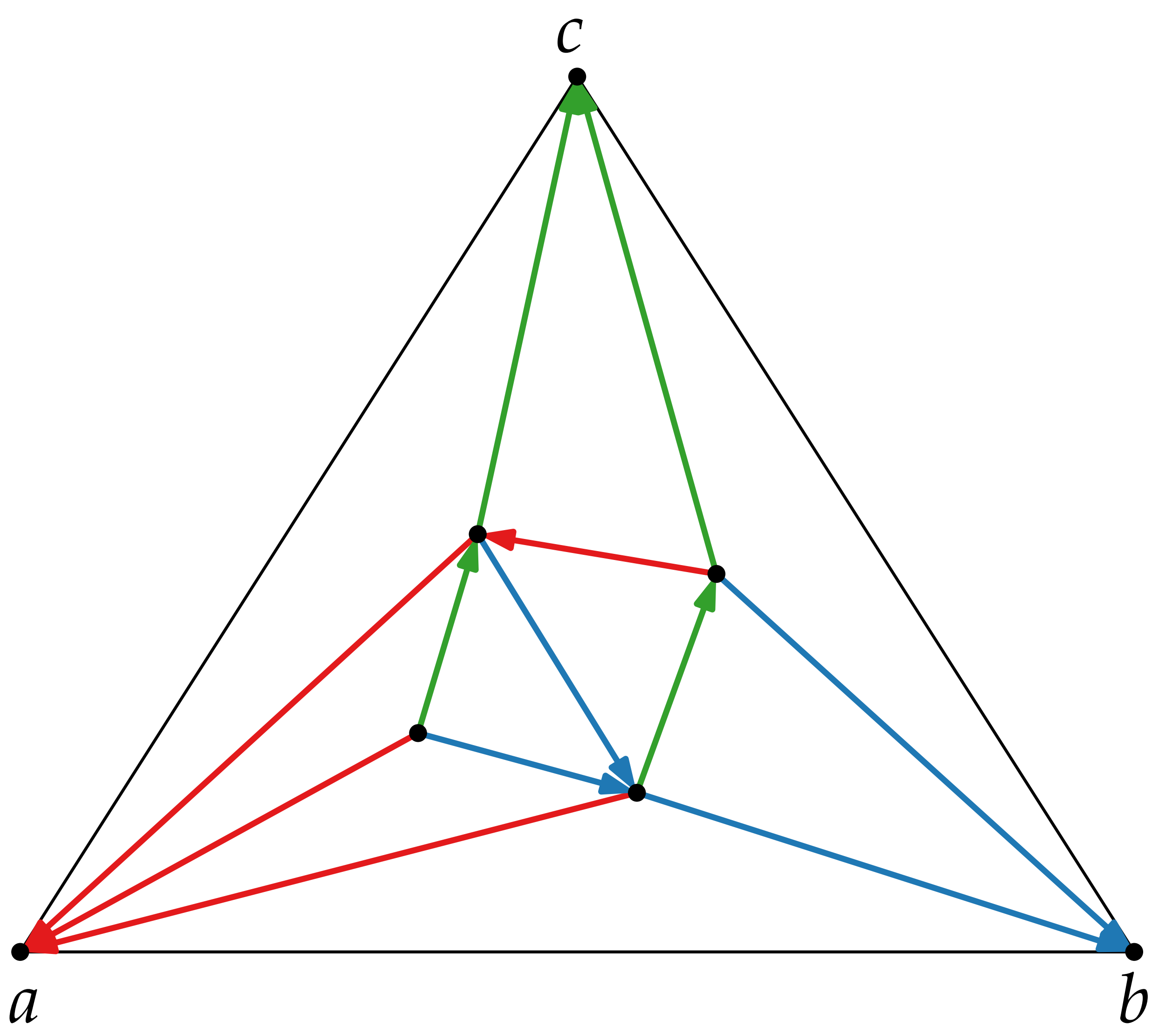

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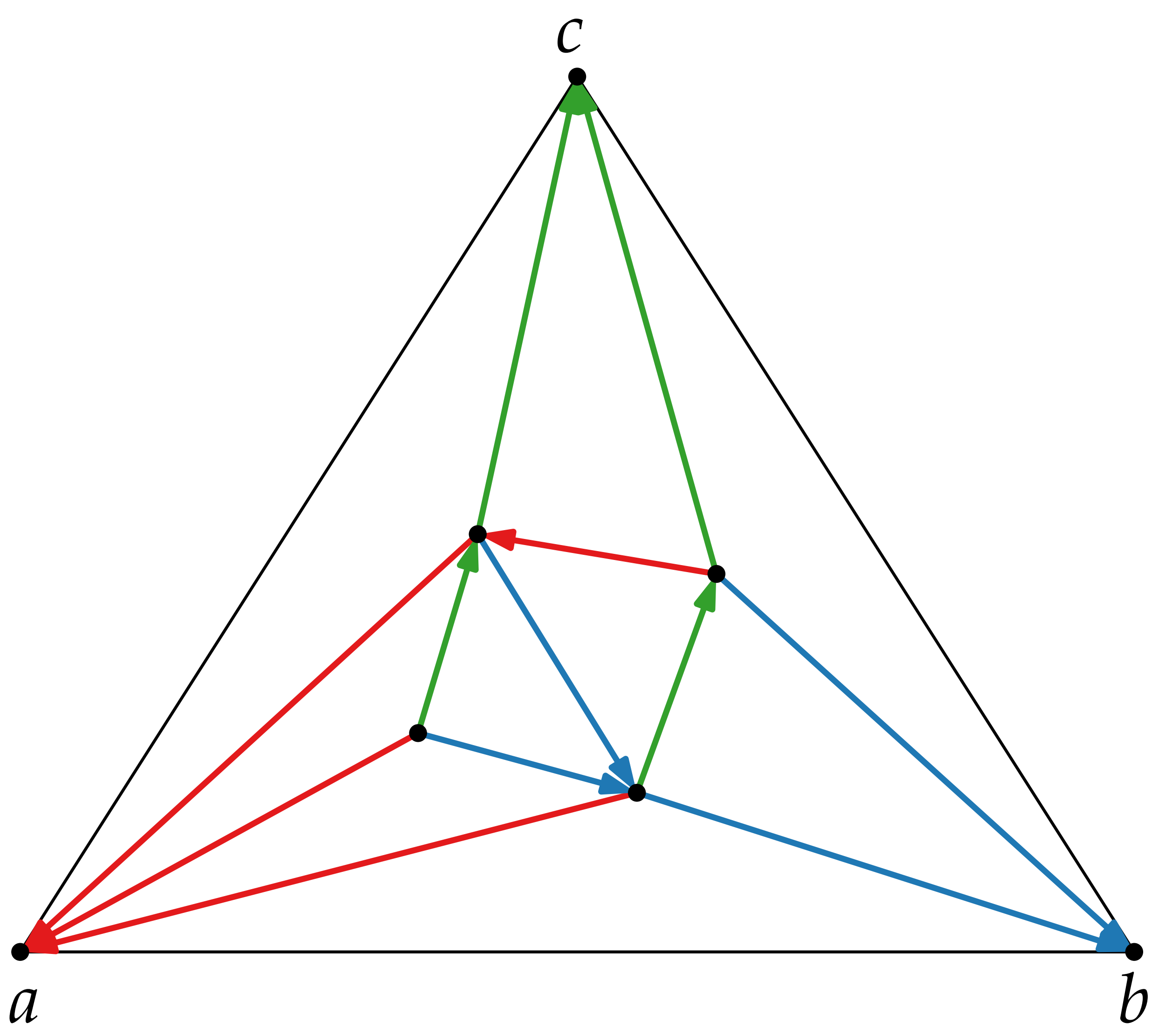

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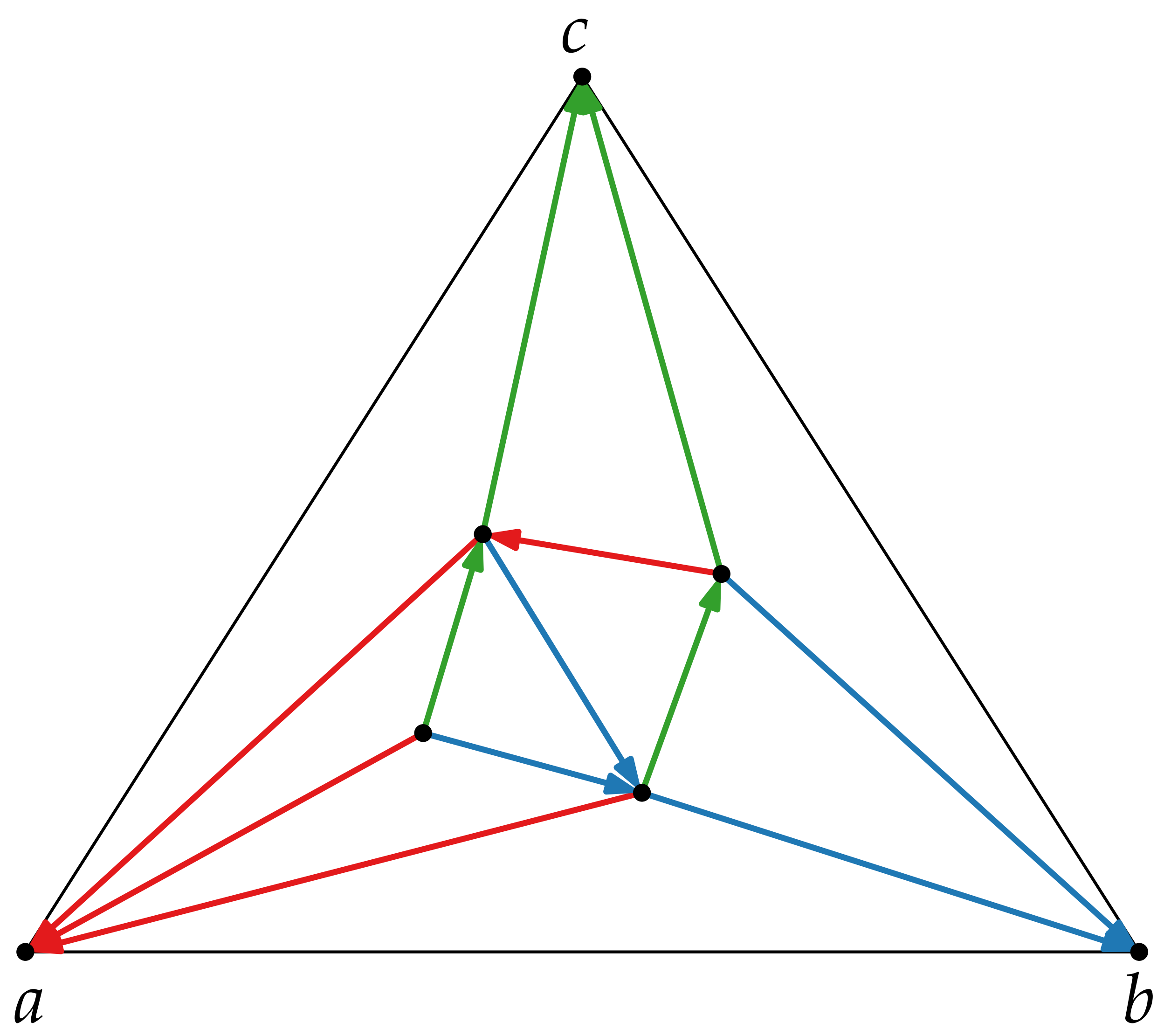

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This is ensured by construction via contraction operation.
(Bonus: Can construct all valid Schnyder realiser.)

Schnyder realiser – canonical order

Adding v_{k+1} to graph G_k

- $v_{k+1}w_p \in T_1$
- $v_{k+1}w_q \in T_2$
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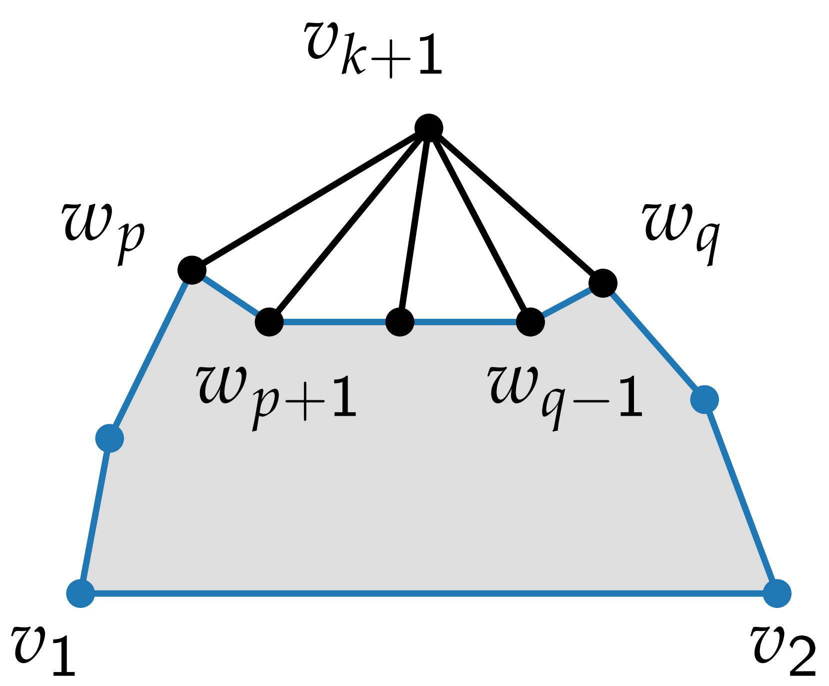

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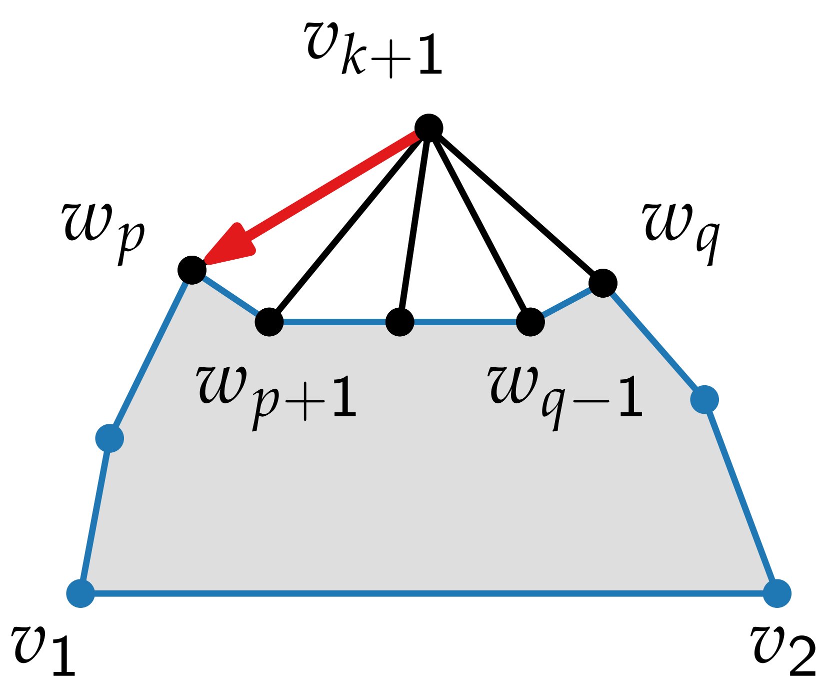

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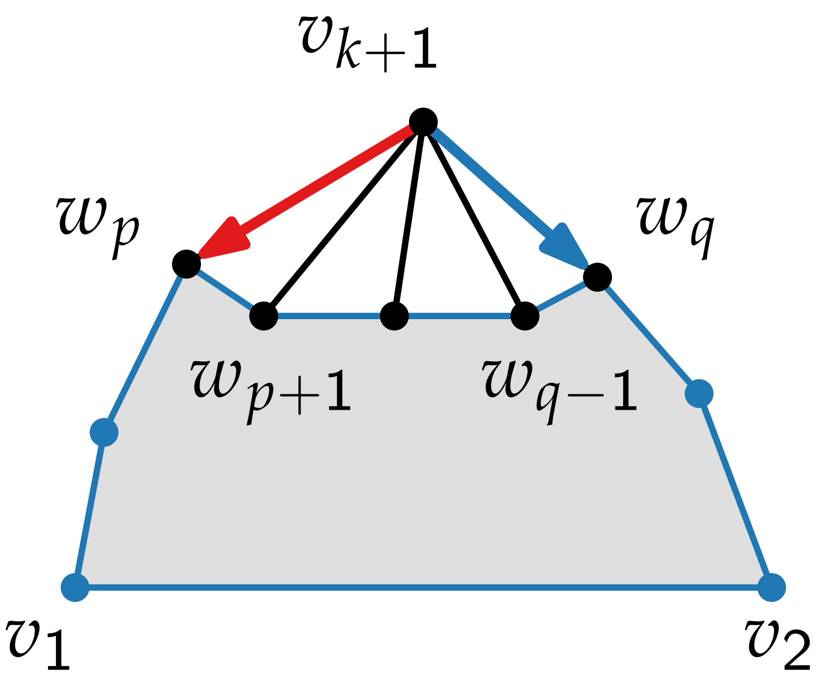

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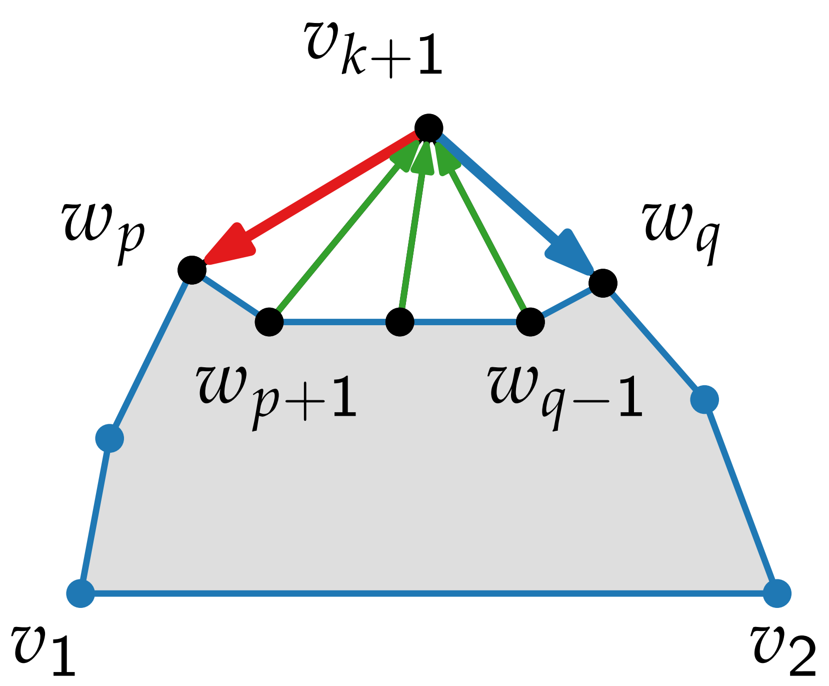

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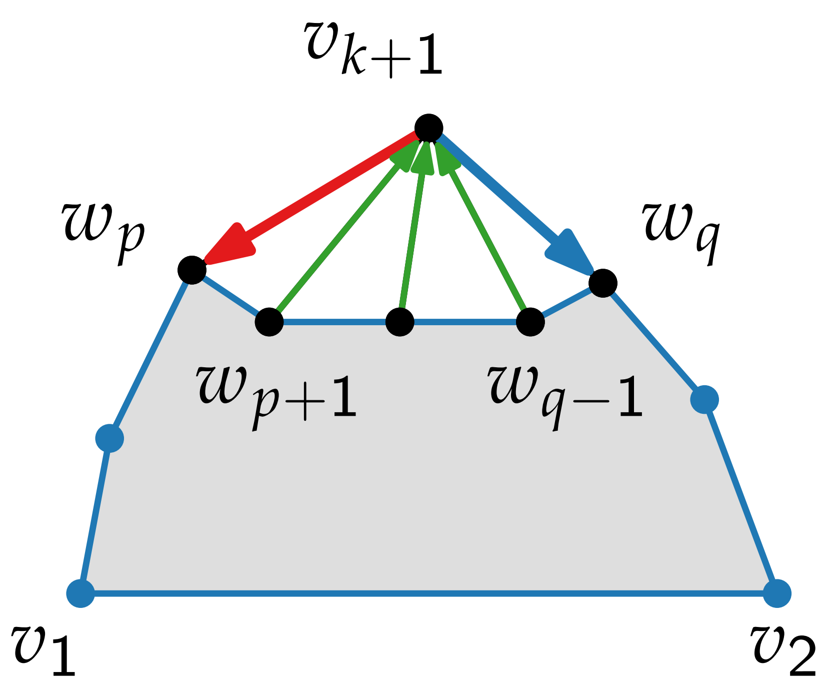

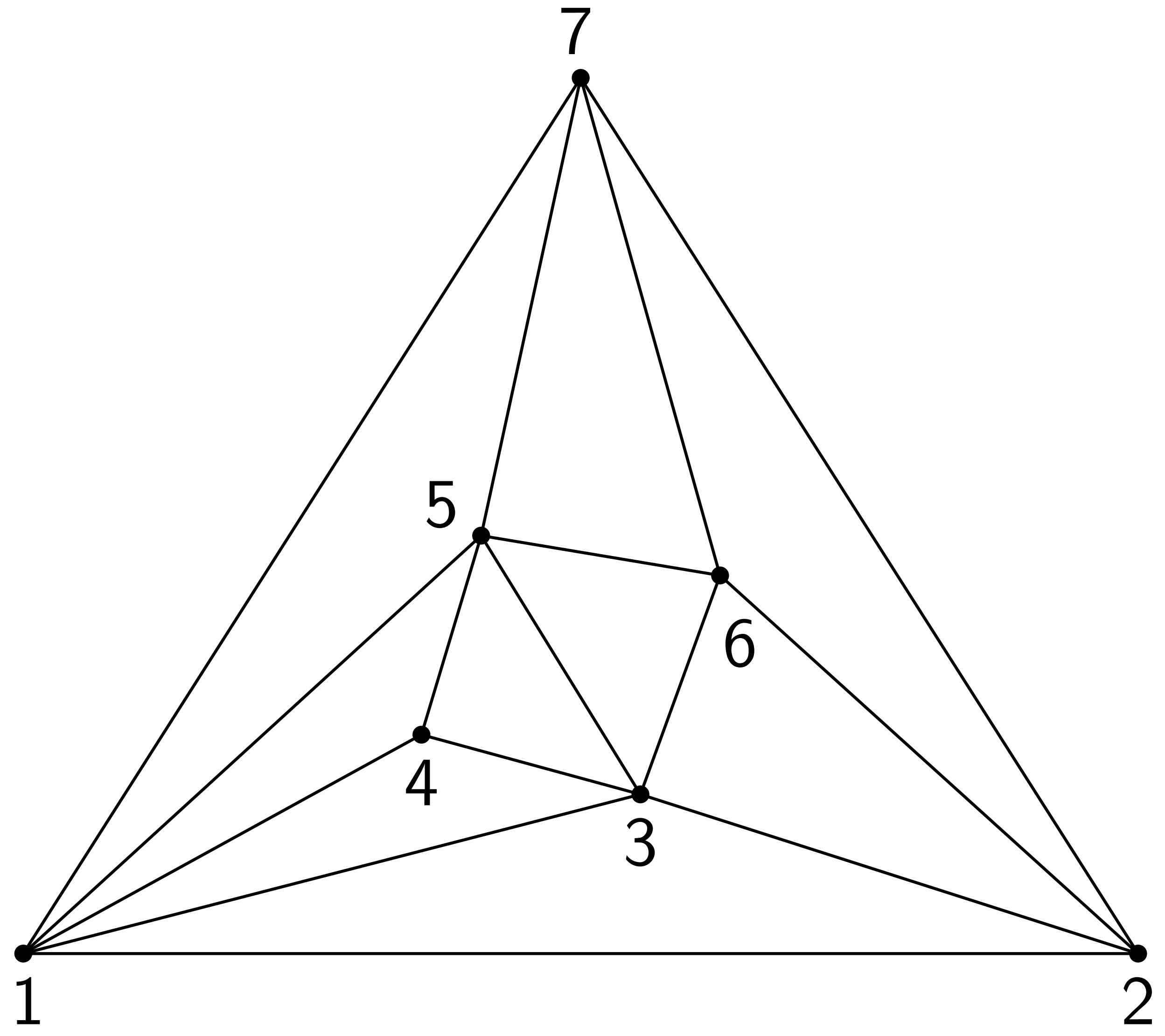

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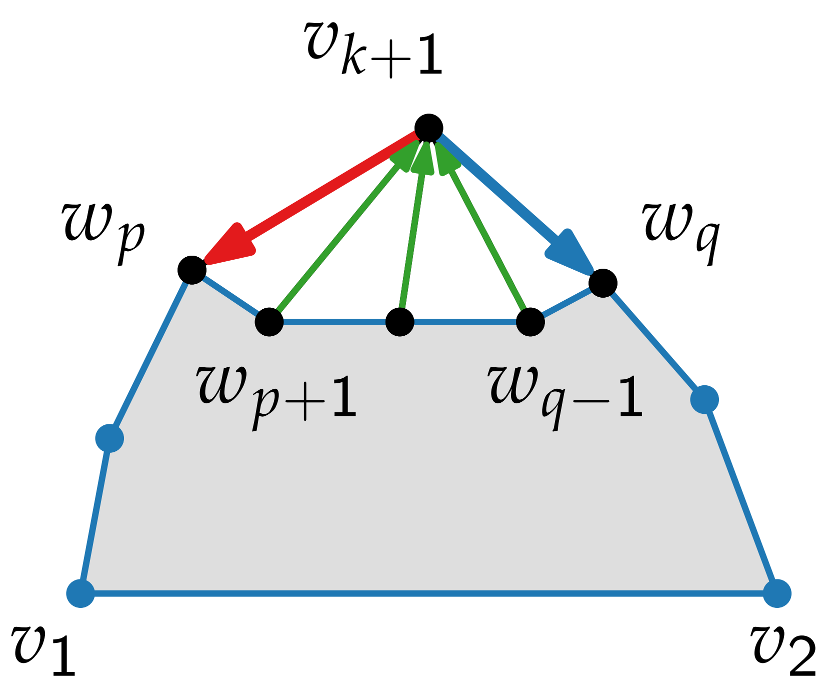

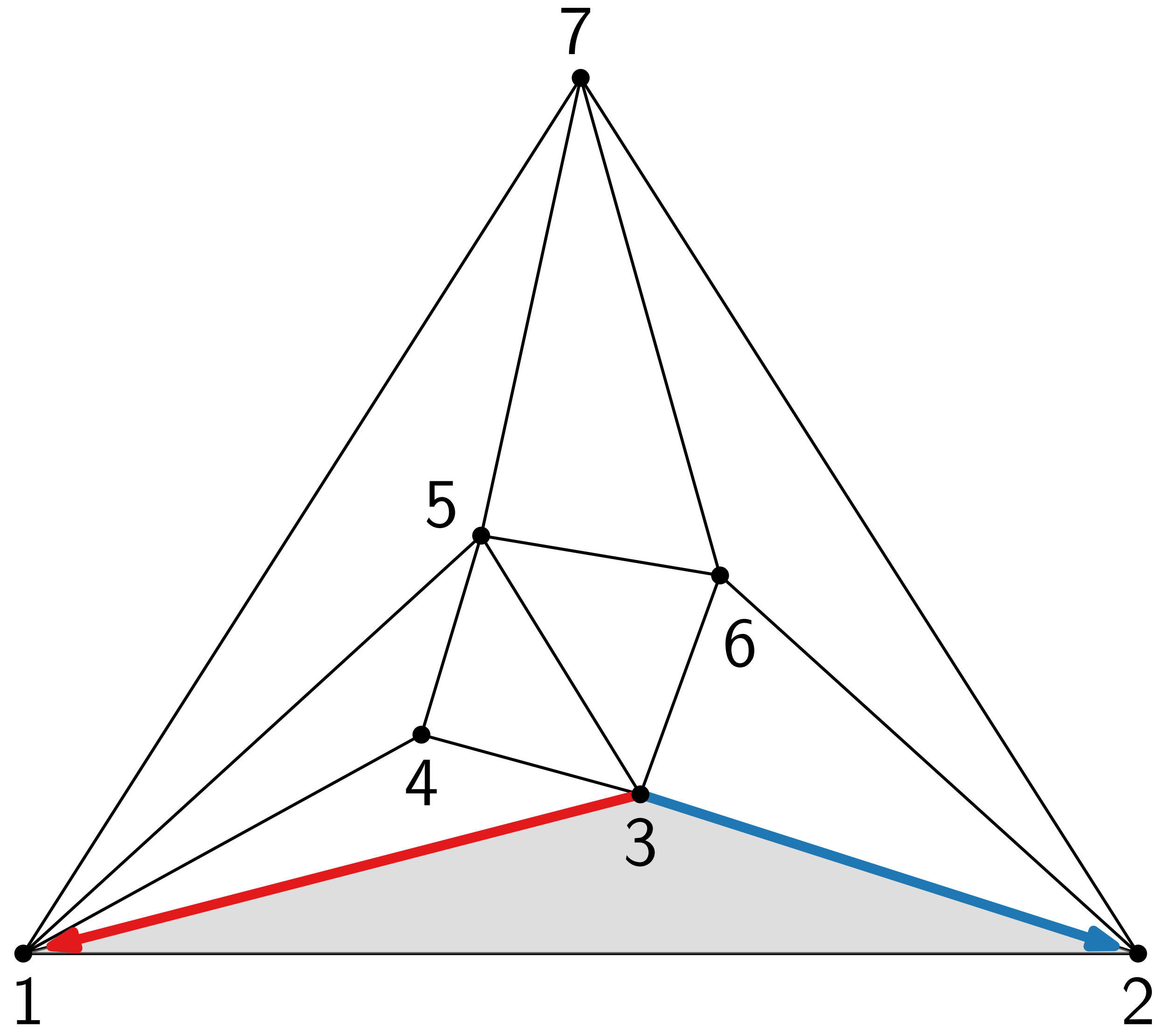

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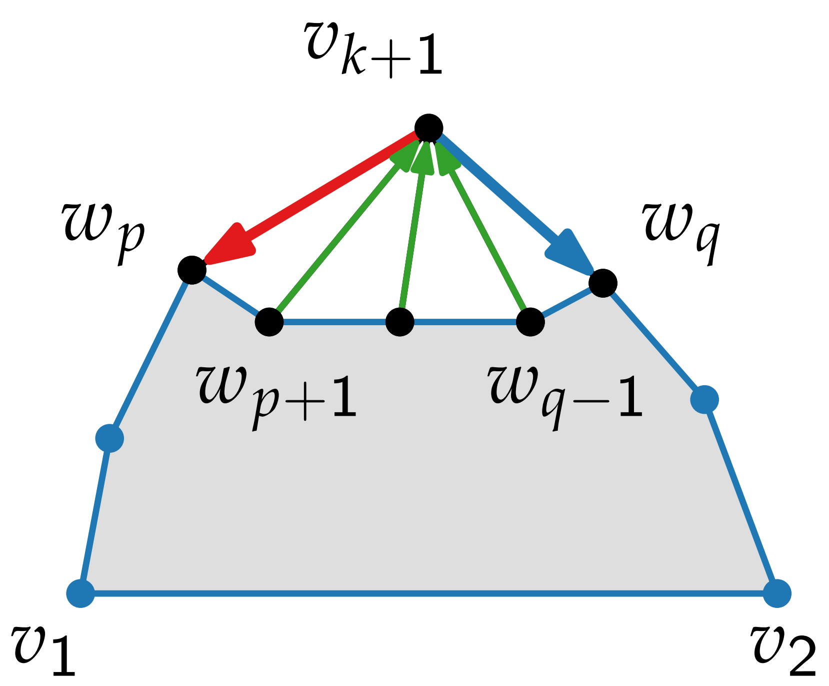

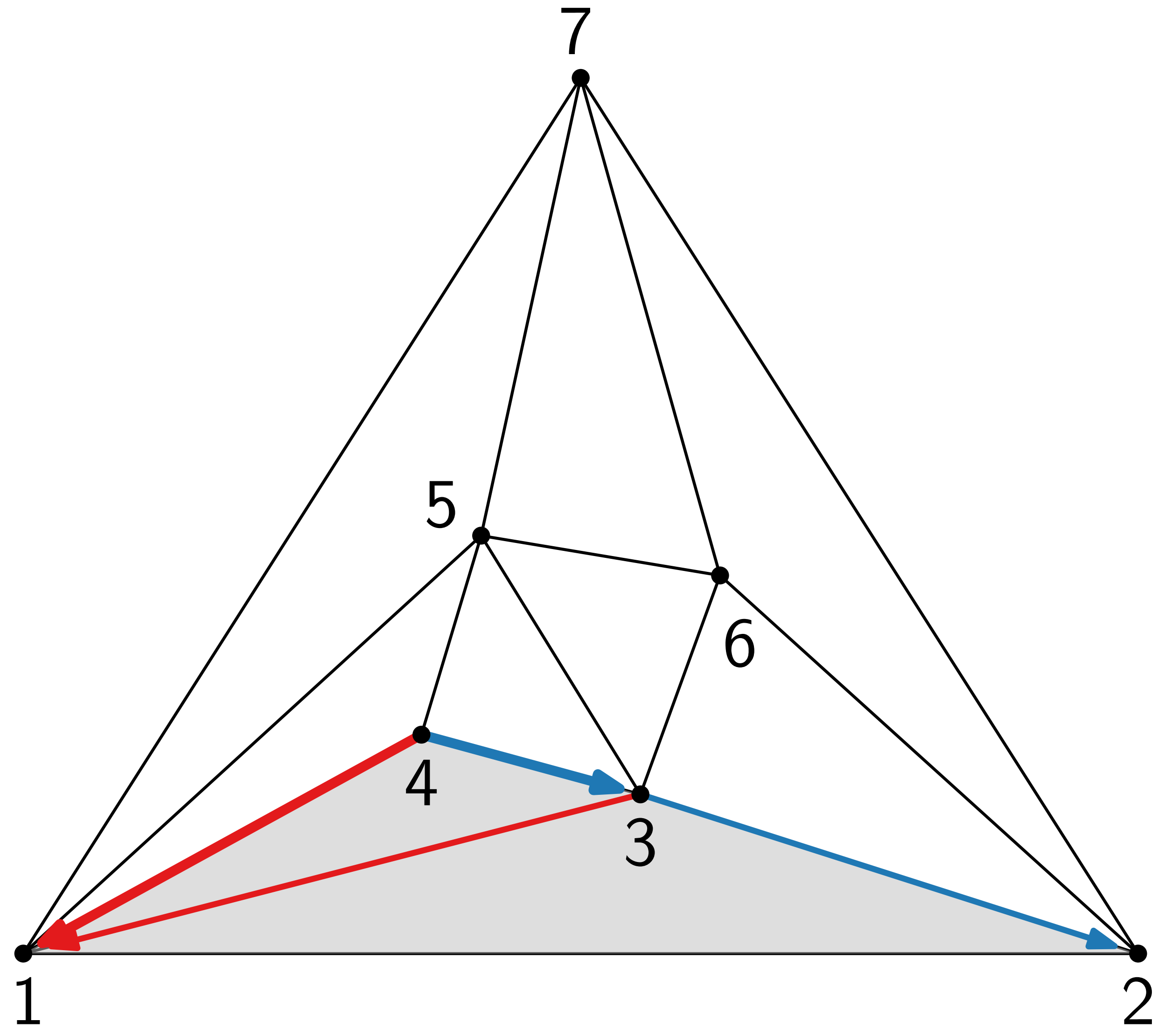

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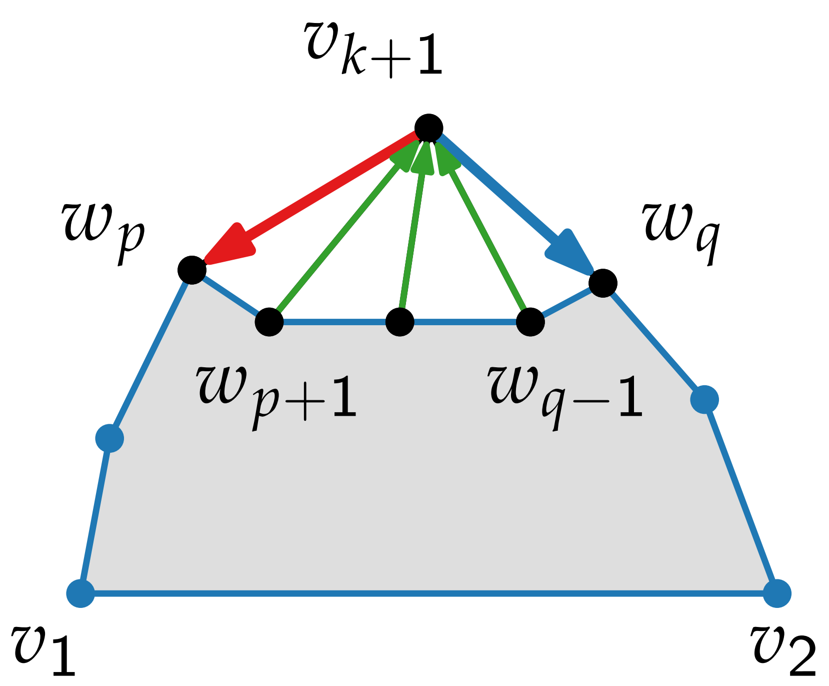

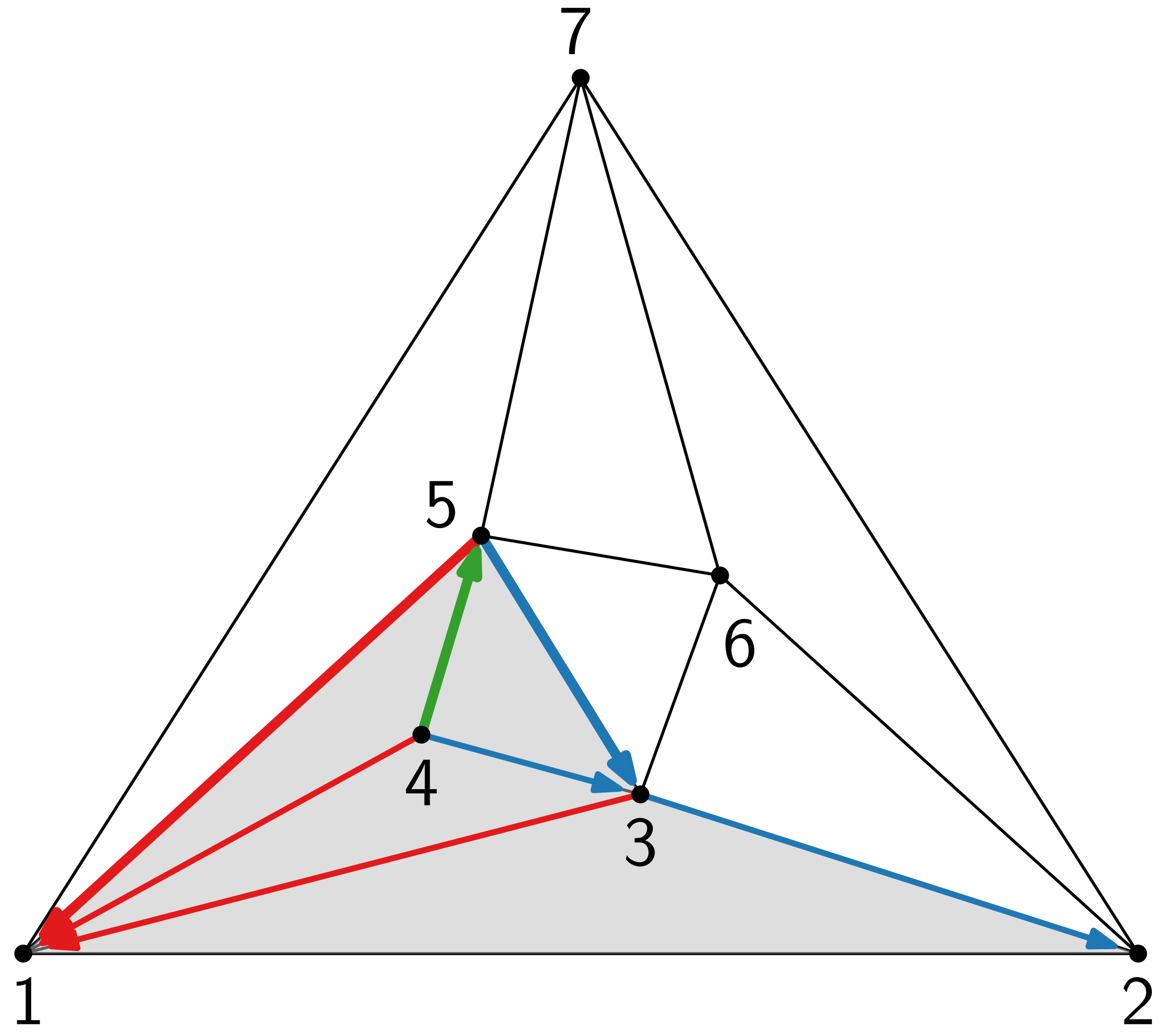

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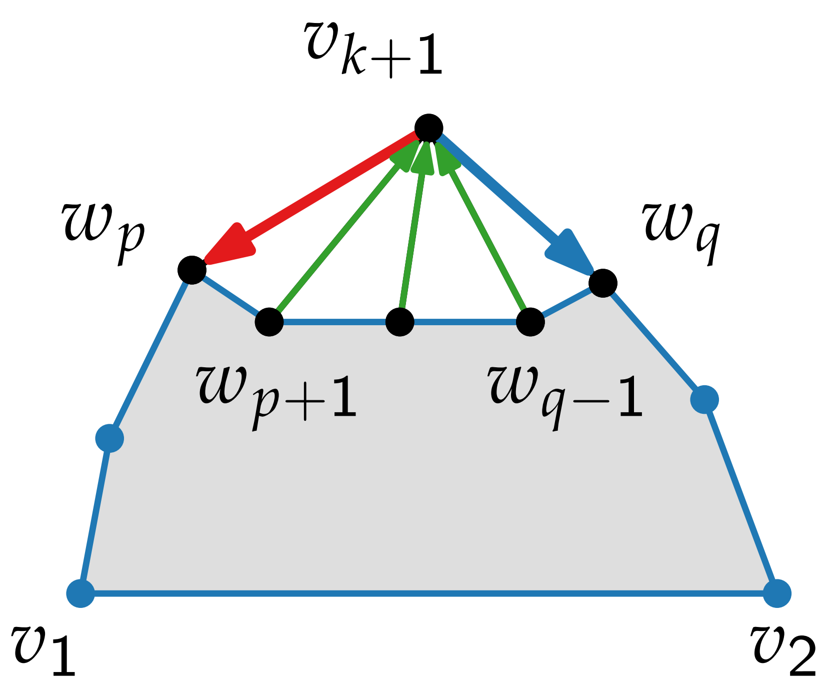

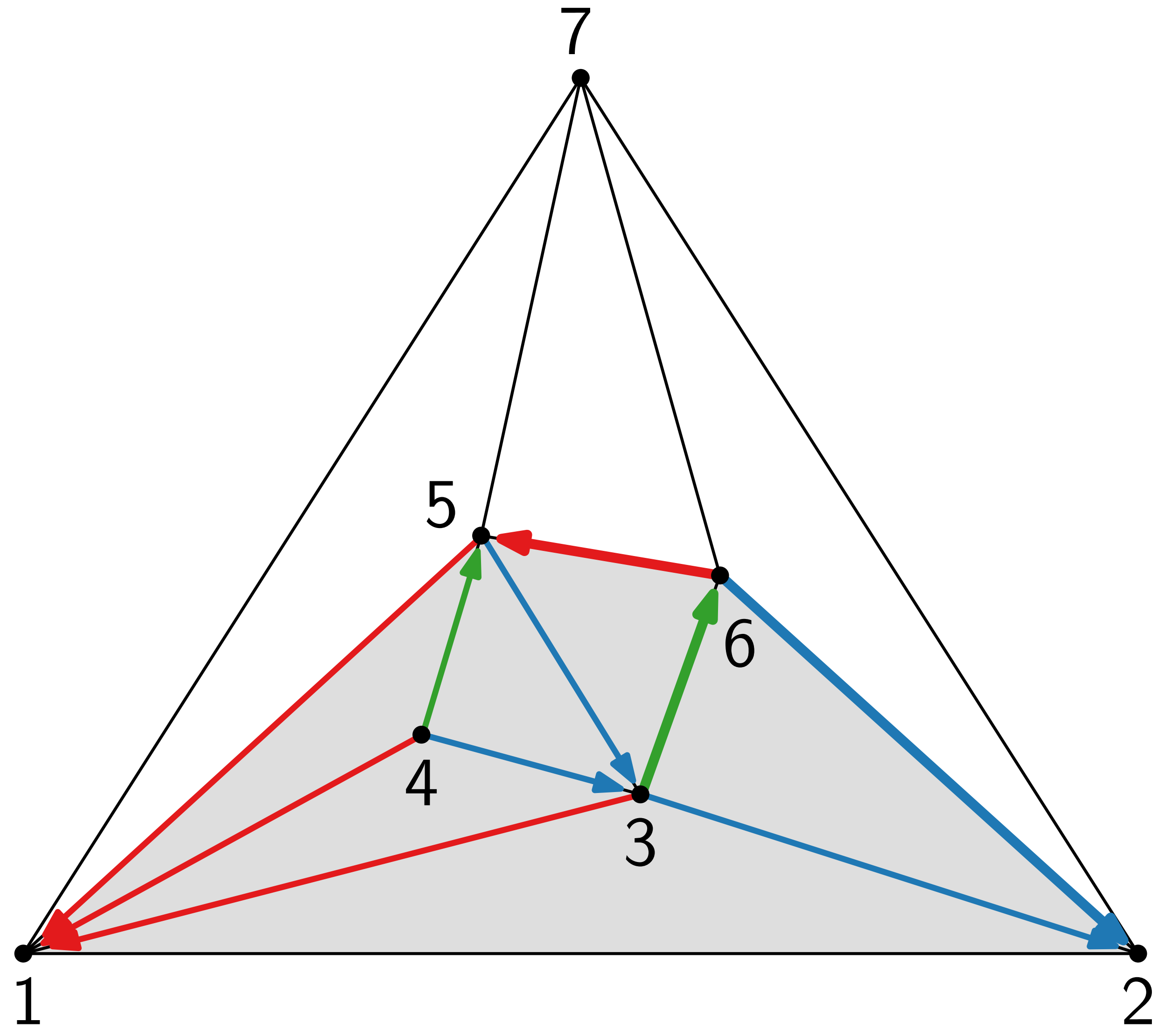

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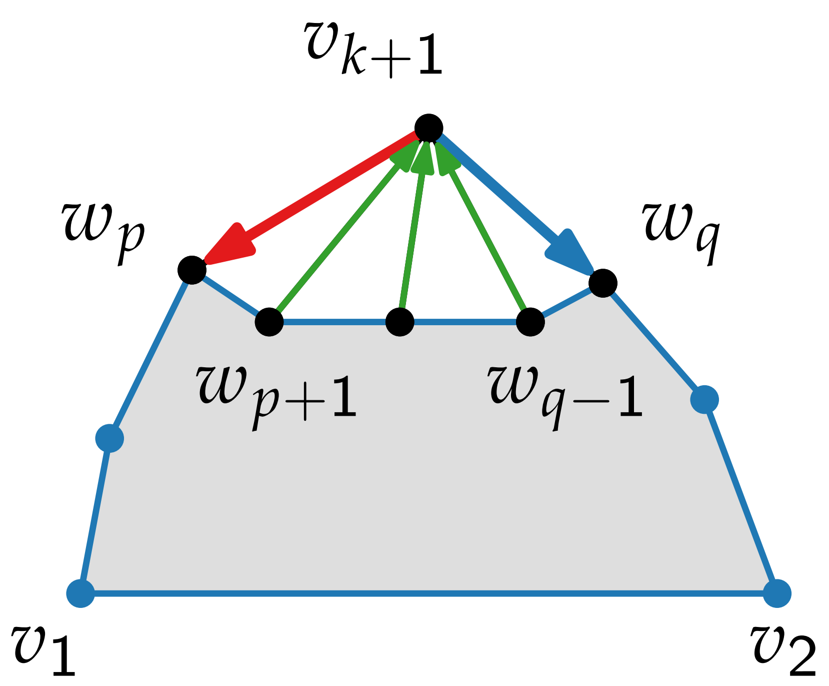

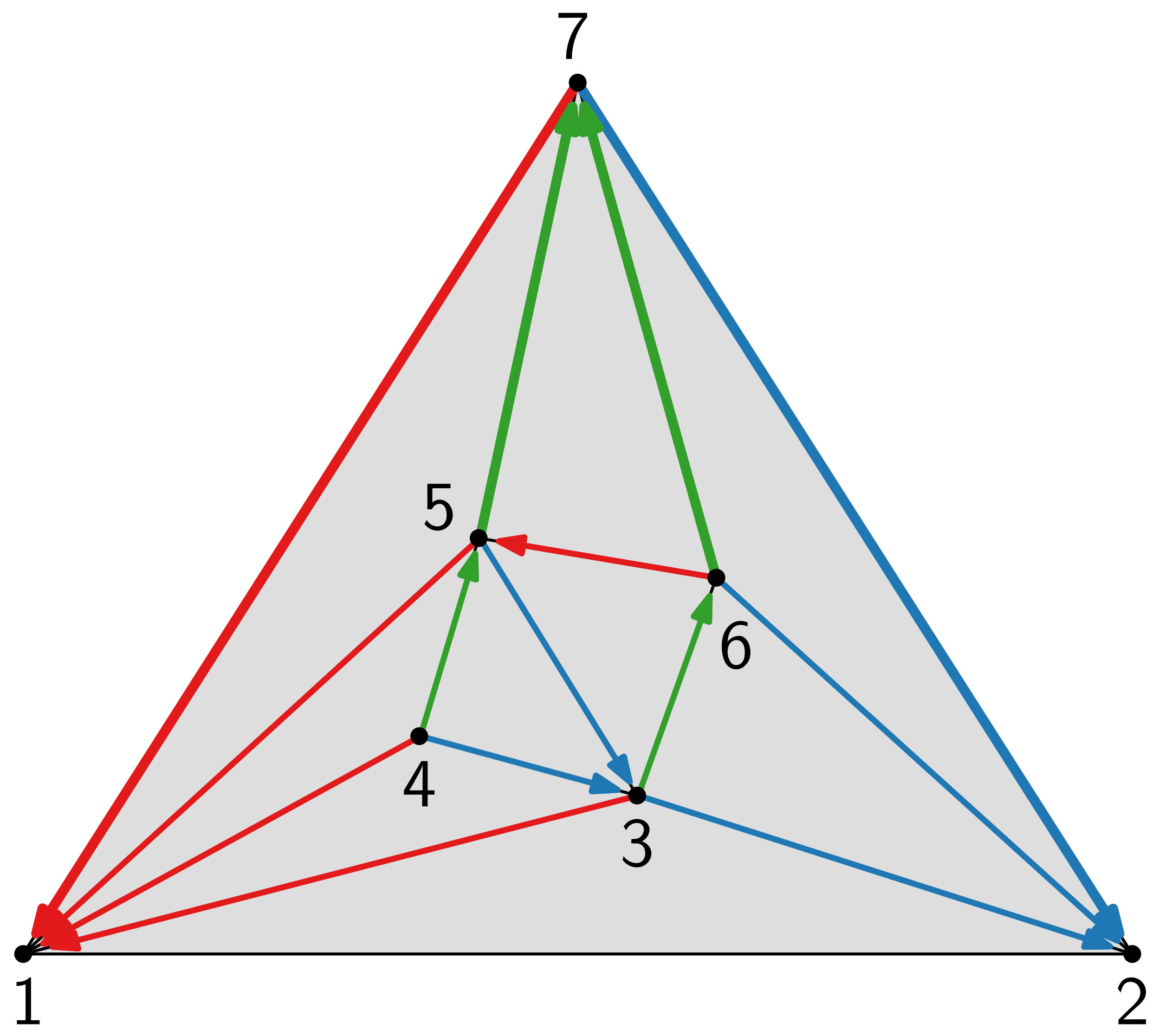

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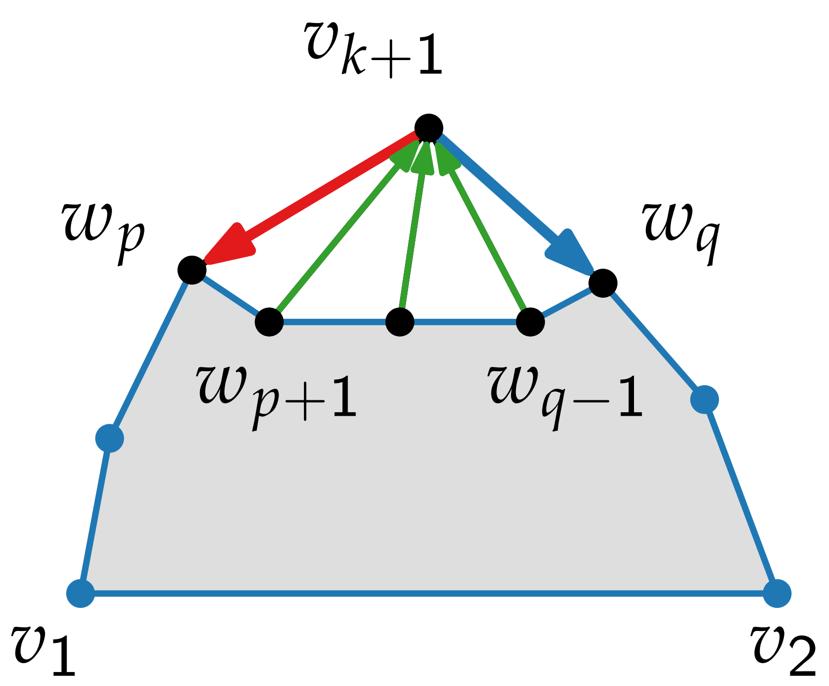
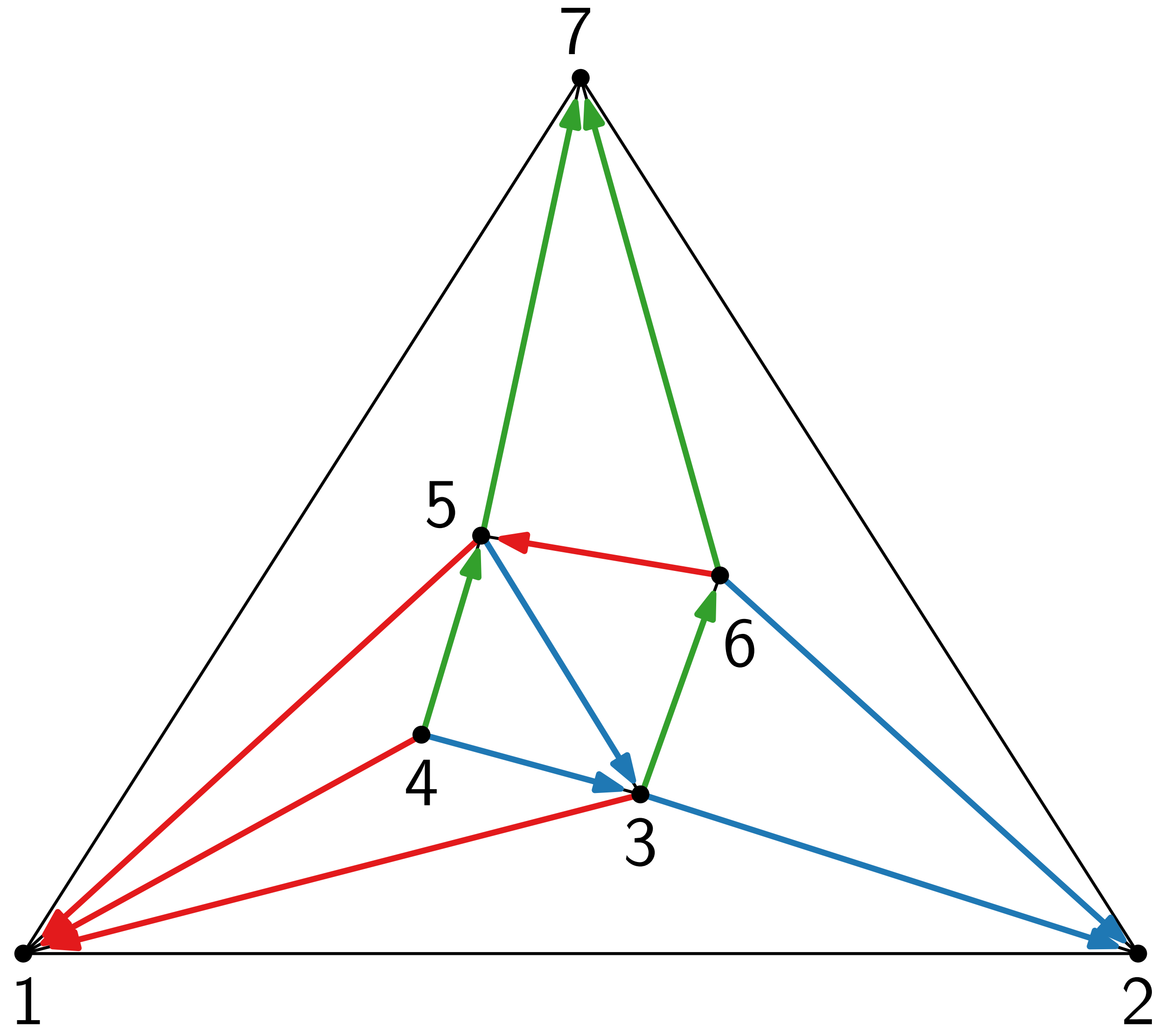
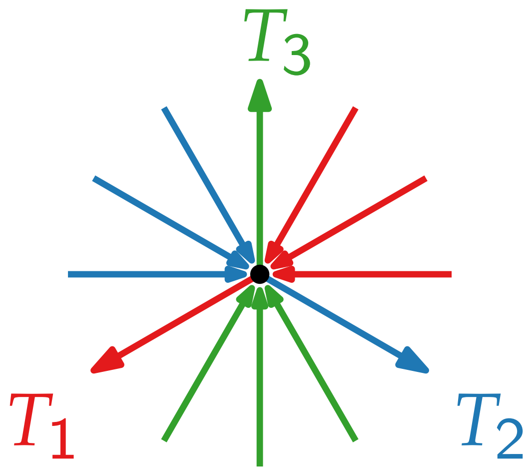




Schnyder drawing

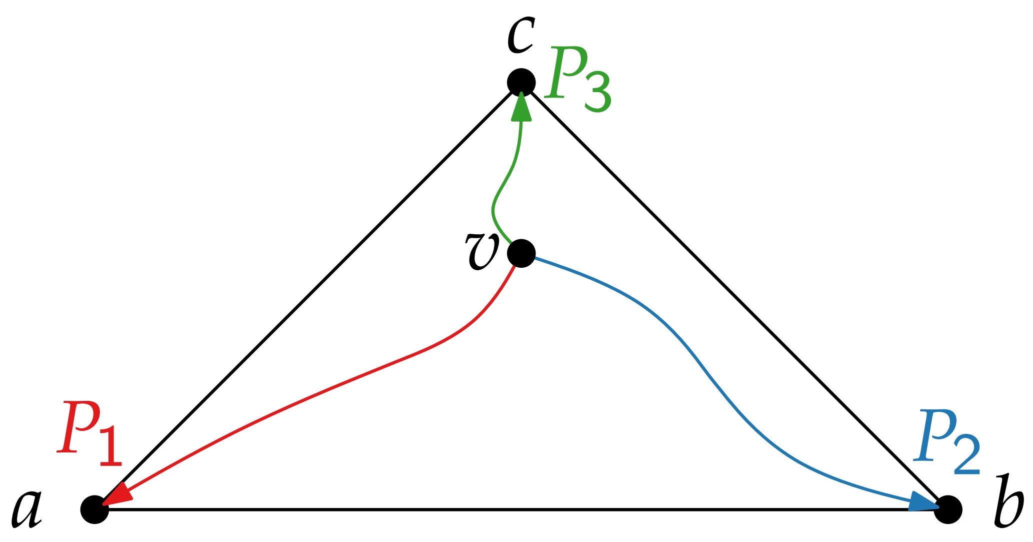
- How to get from Schnyder realiser to barycentric representation



$$f : v \in V \mapsto v_1 A + v_2 B + v_3 C$$

Face regions

- $P_i(v)$ path from v to source of T_i



Face regions

- $P_i(v)$ path from v to source of T_i
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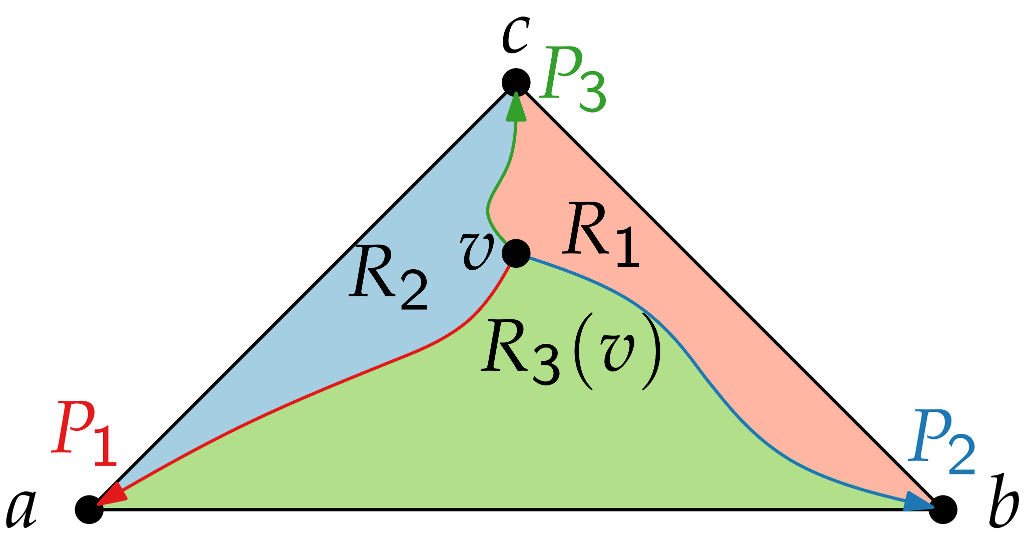

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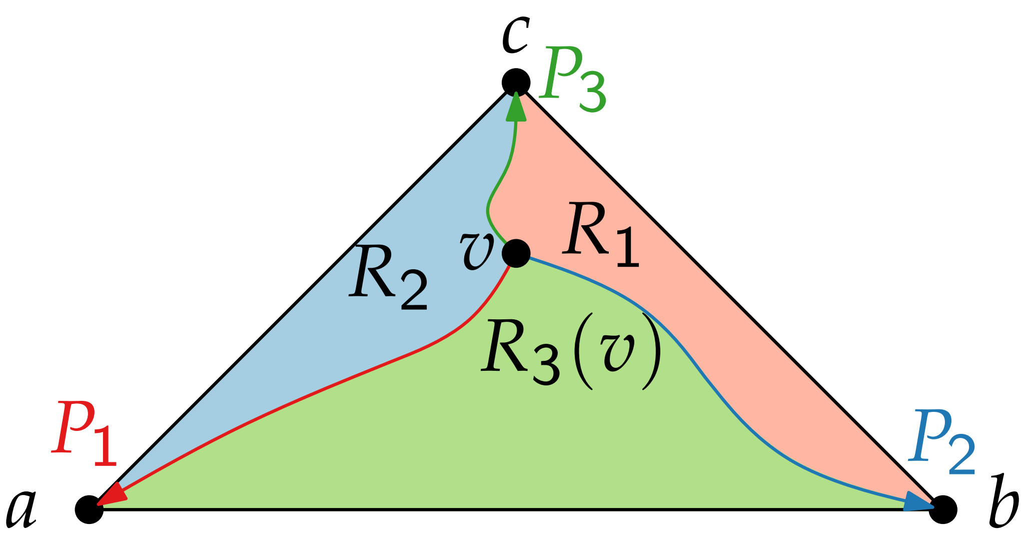
- Paths $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at vertex v.



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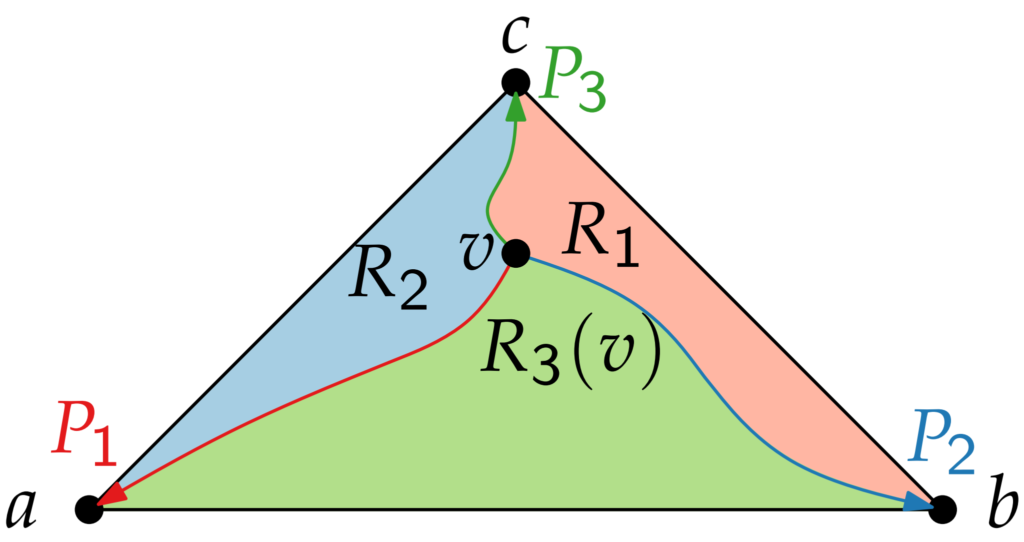
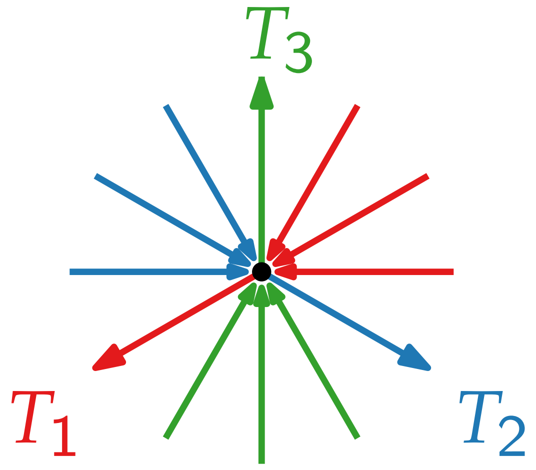


Proof . . .



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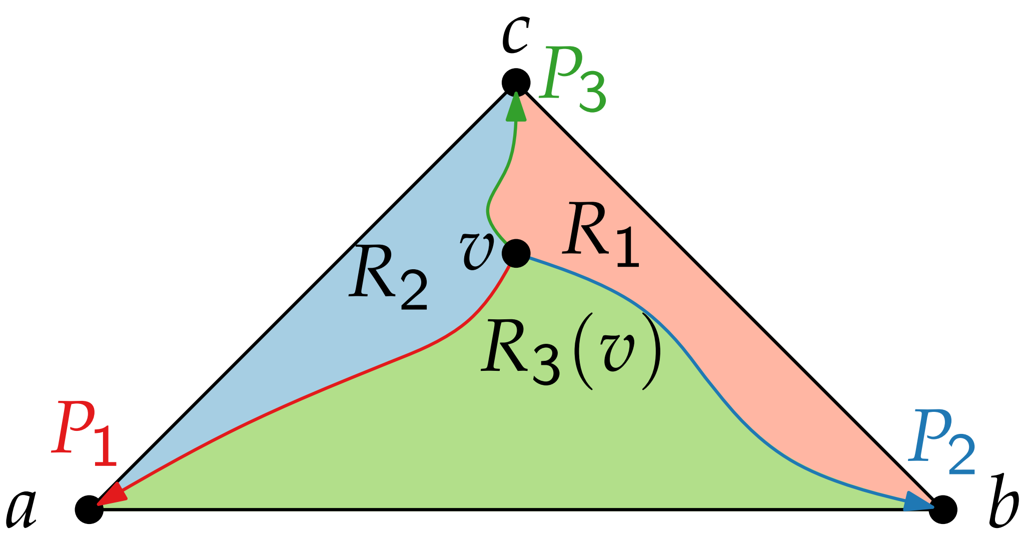

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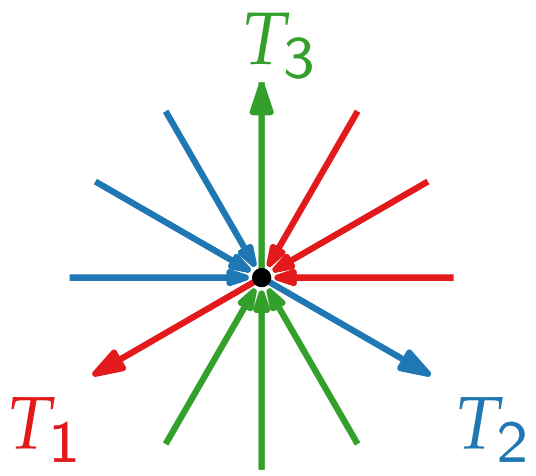

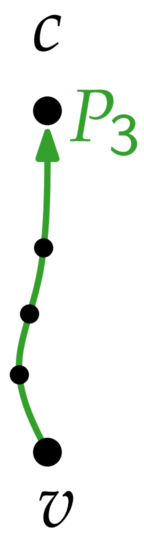

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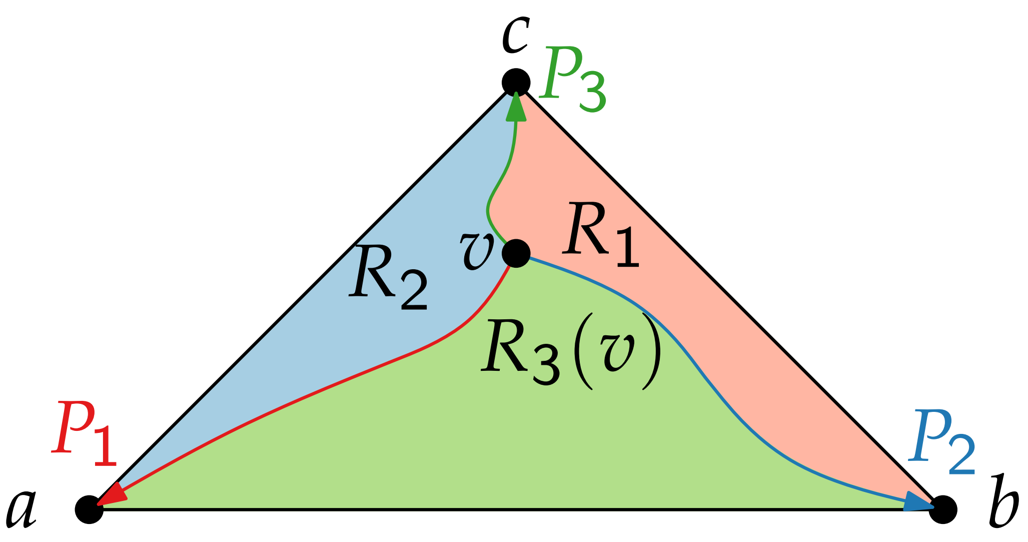

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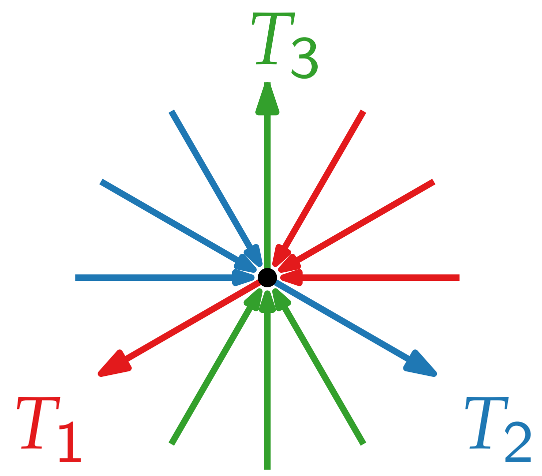

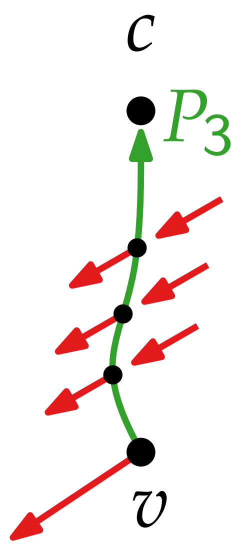

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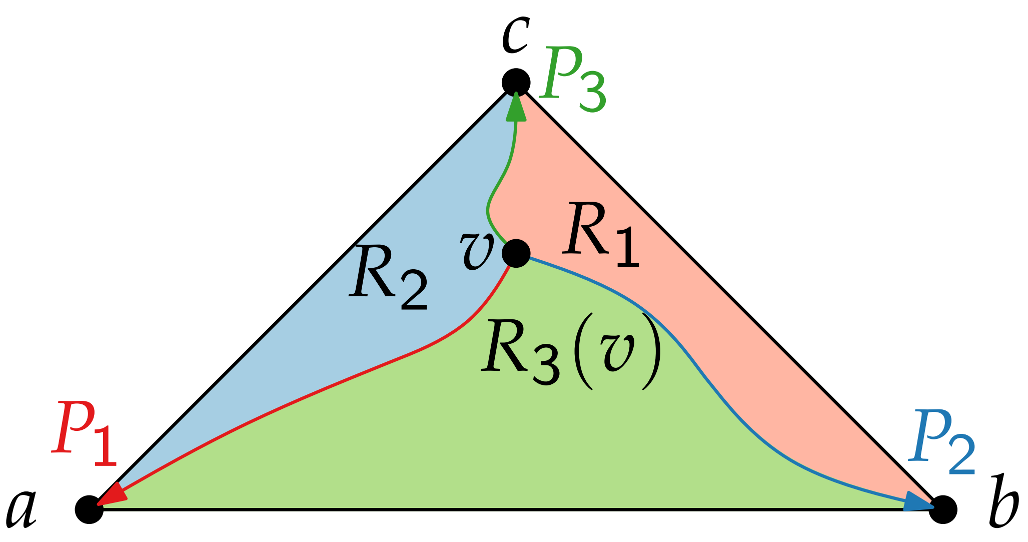

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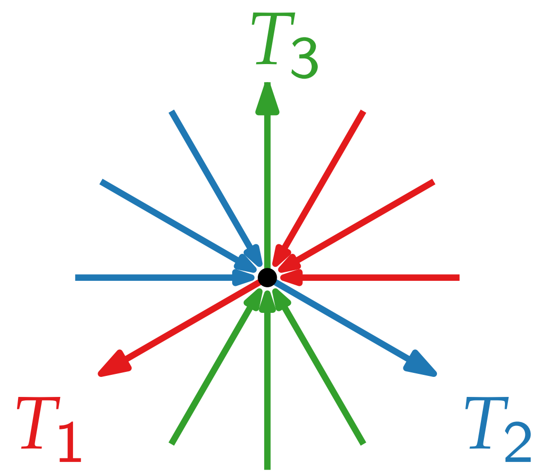

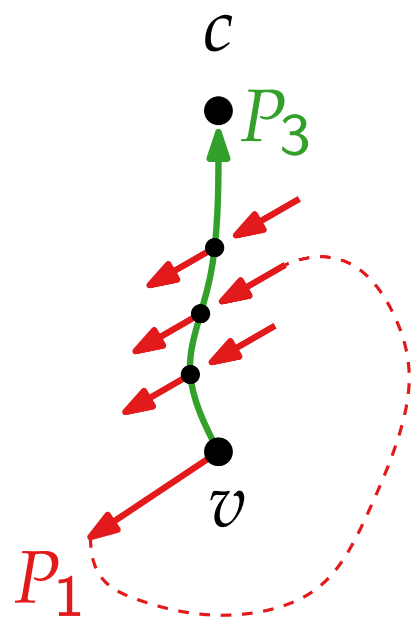

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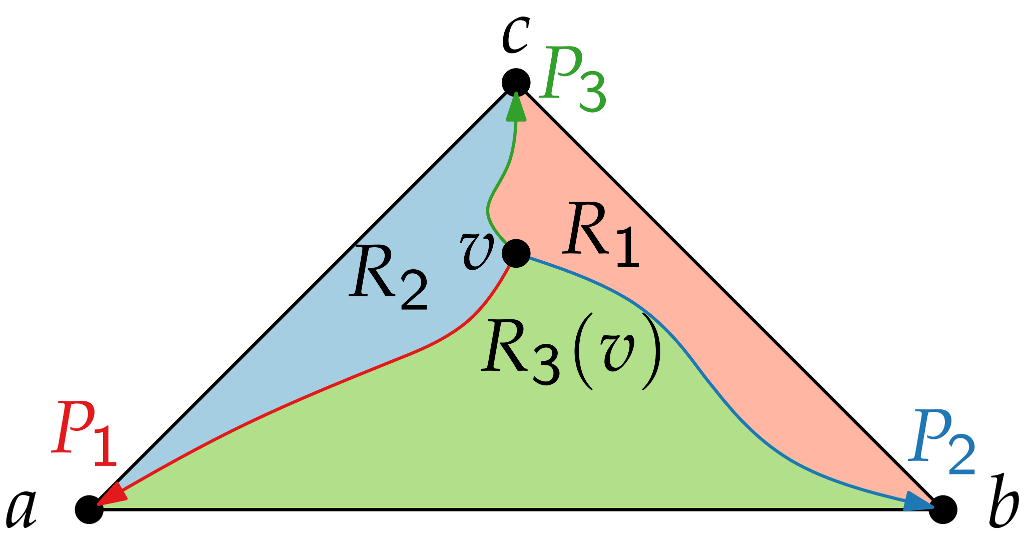

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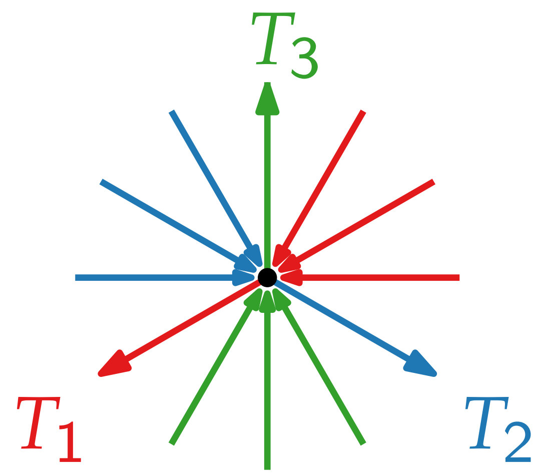

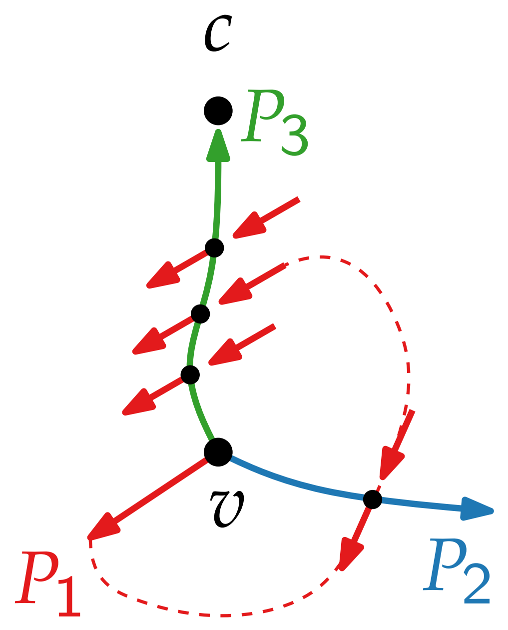

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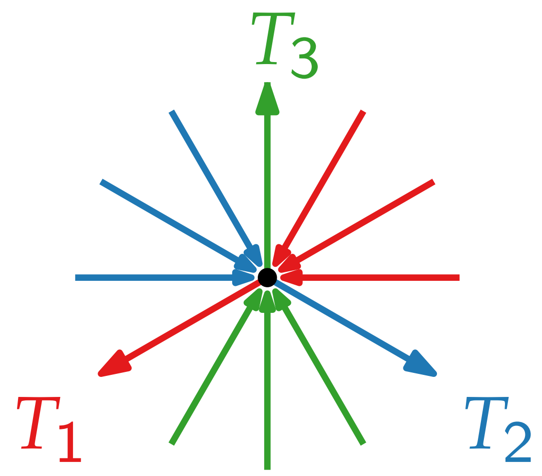

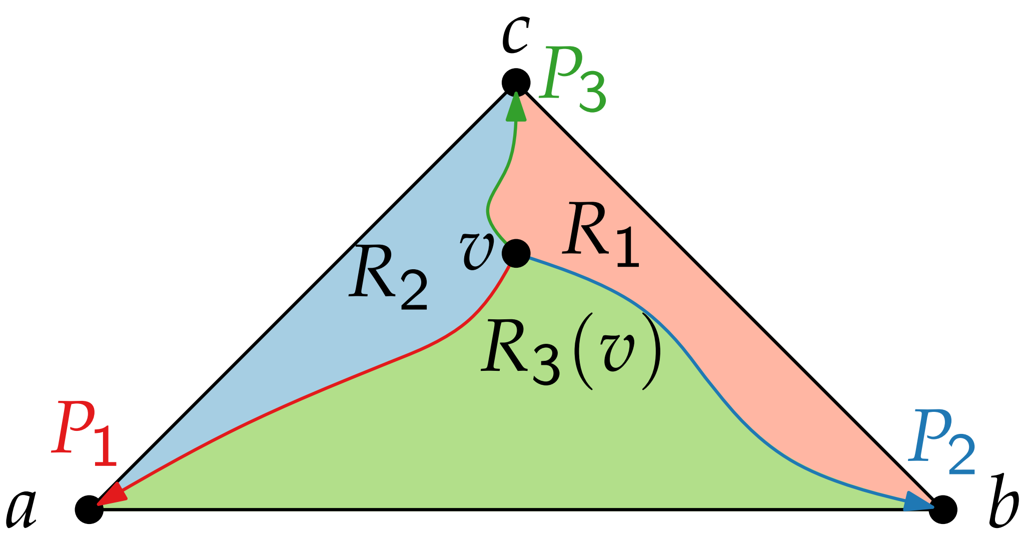

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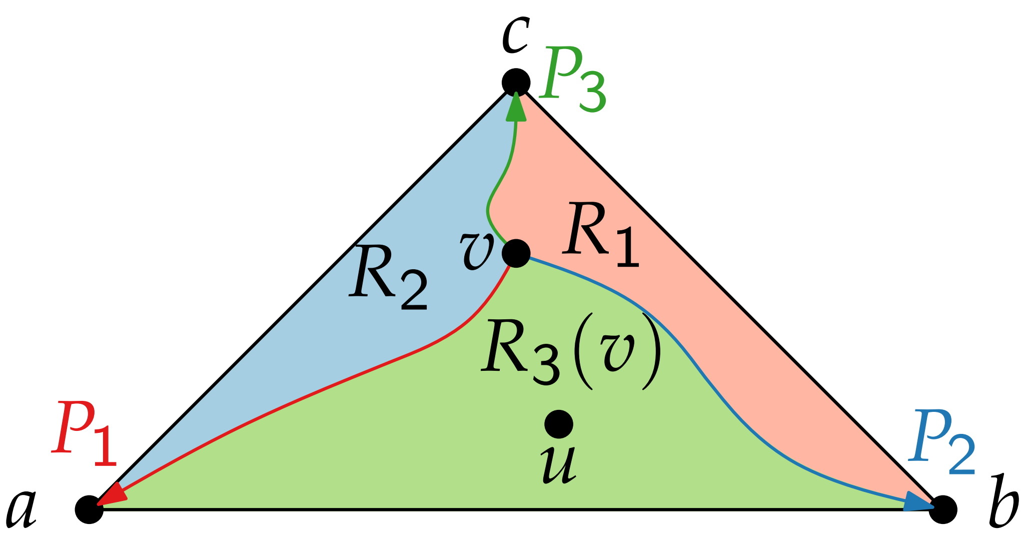

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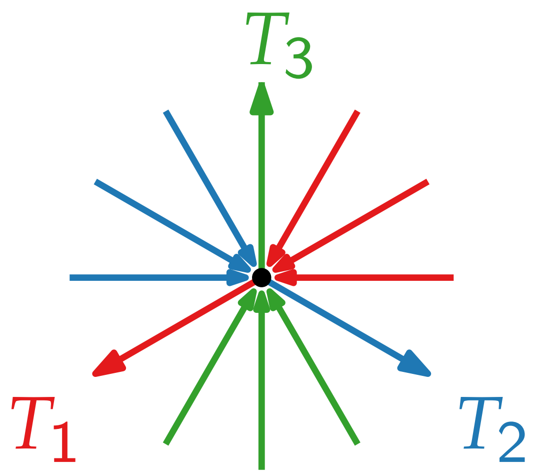

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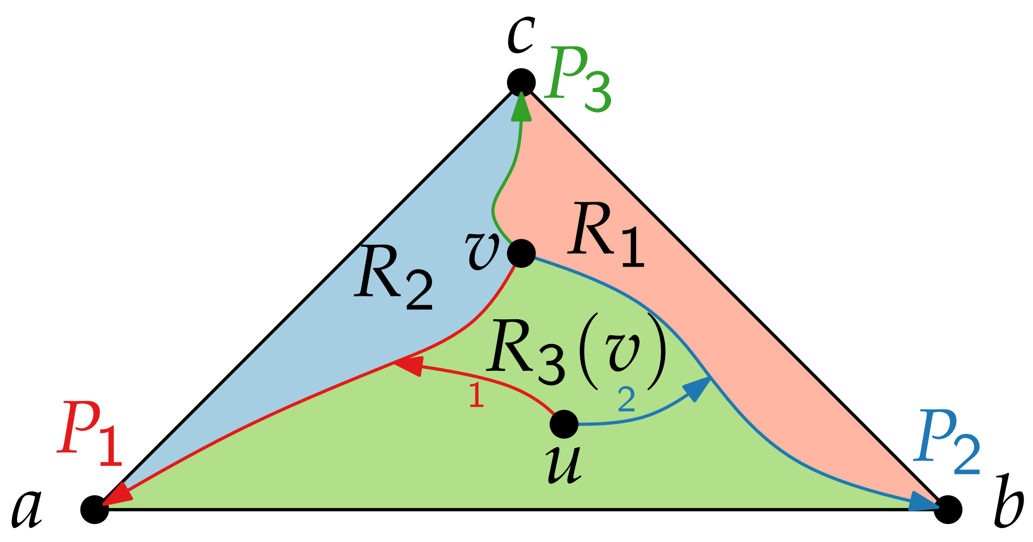

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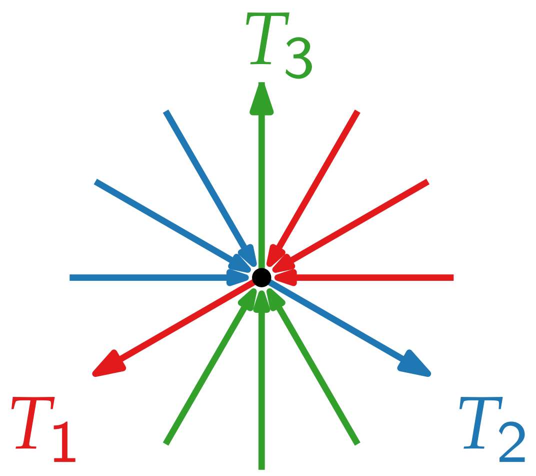

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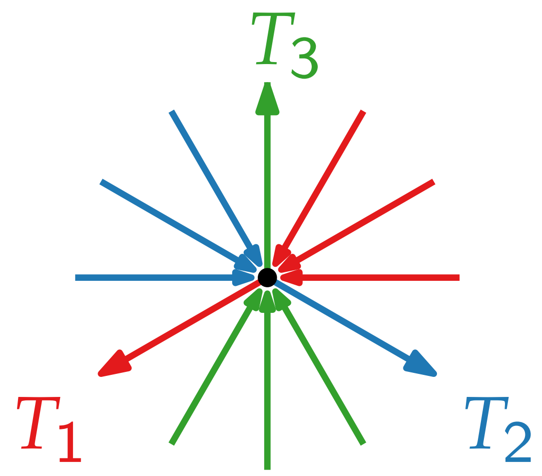

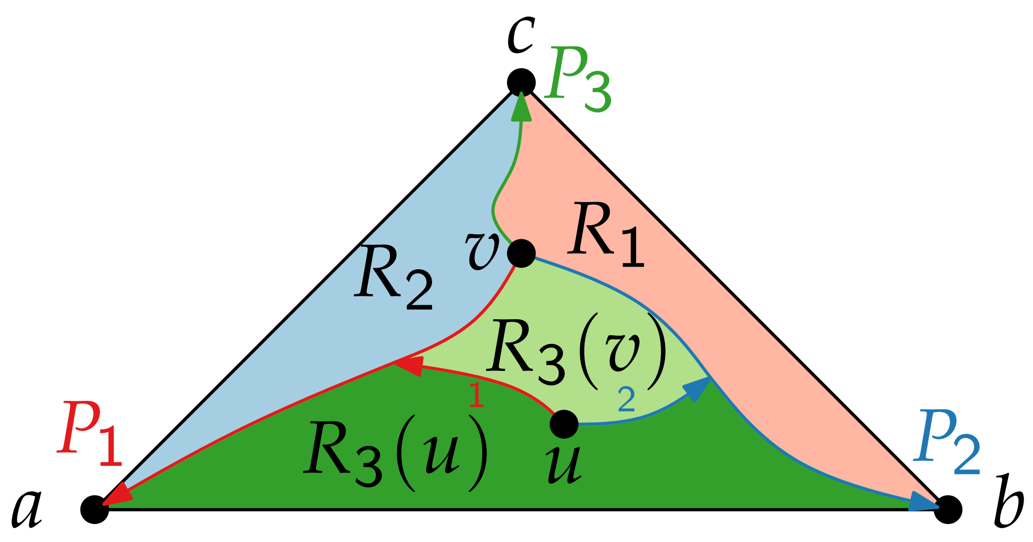

Schnyder drawing

- Let barycentric coordinates of $v \in G \setminus \{a, b, c\}$ be (v_1, v_2, v_3), where $v_1 = |R_1(v)|/(2n-5)$, $v_2 = |R_2(v)|/(2n-5)$ and $v_3 = |R_3(v)|/(2n-5)$.

- Set
  - $A = (2n-5, 0)$
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Theorem.
The mapping

$$f\colon v \mapsto (v_1, v_2, v_3) = \tfrac{1}{2n-5}(|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G, which thus gives a planar straight-line drawing of G in a $(2n-5) \times (2n-5)$ grid.

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Proof.
- Condition 1: $v_1 + v_2 + v_3 = 1$

Schnyder drawing

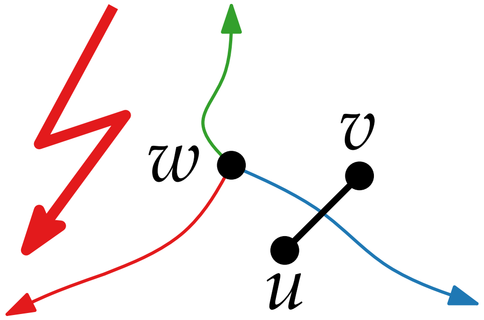
- Let barycentric coordinates of $v \in G \setminus \{a, b, c\}$ be (v_1, v_2, v_3), where $v_1 = |R_1(v)|/(2n-5)$, $v_2 = |R_2(v)|/(2n-5)$ and $v_3 = |R_3(v)|/(2n-5)$.

- Set
  - $A = (2n-5, 0)$
  - $B = (0, 2n-5)$
  - $C = (0, 0)$

Theorem.
The mapping

$$f \colon v \mapsto (v_1, v_2, v_3) = \frac{1}{2n-5}(|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G, which thus gives a planar straight-line drawing of G in a $(2n-5) \times (2n-5)$ grid.



Proof.

- Condition 1: $v_1 + v_2 + v_3 = 1$
- Condition 2: For each edge $\{u, v\}$ and vertex $w \neq u, v$ at least one of three is true: $w_1 > u_1, v_1$, $w_2 > u_2, v_2$, $w_3 > u_3, v_3$.

Weak barycentric representation

Definition.
A **weak barycentric representation** of a graph $G = (V, E)$ is an *injective* map $v \in V \mapsto (v_1, v_2, v_3) \in \mathbb{R}^3$ with the following properties:

- $v_1 + v_2 + v_3 = 1$ for every $v \in V$
- for every $\{x, y\} \in E$ and every $z \in V \setminus \{x, y\}$ there is $k \in \{1, 2, 3\}$ with $(x_k, x_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$ and $(y_k, y_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$.

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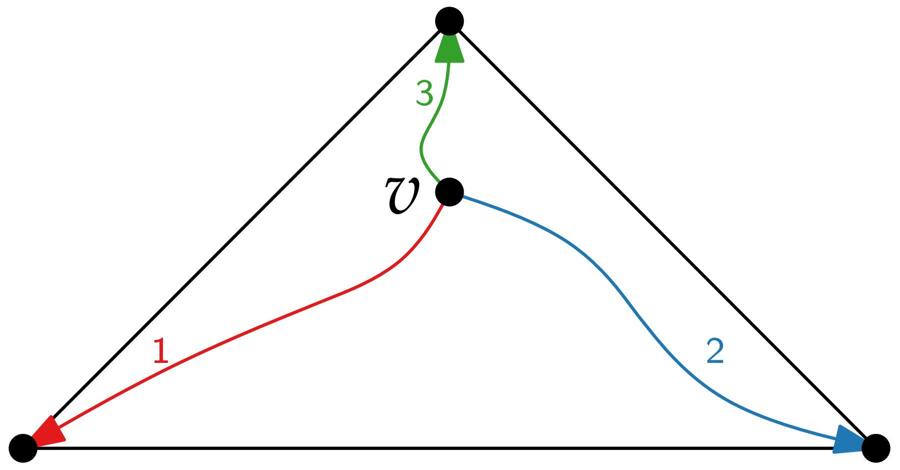
Proof is similar to before.. and thus an exercise.

New barycentric coordinates



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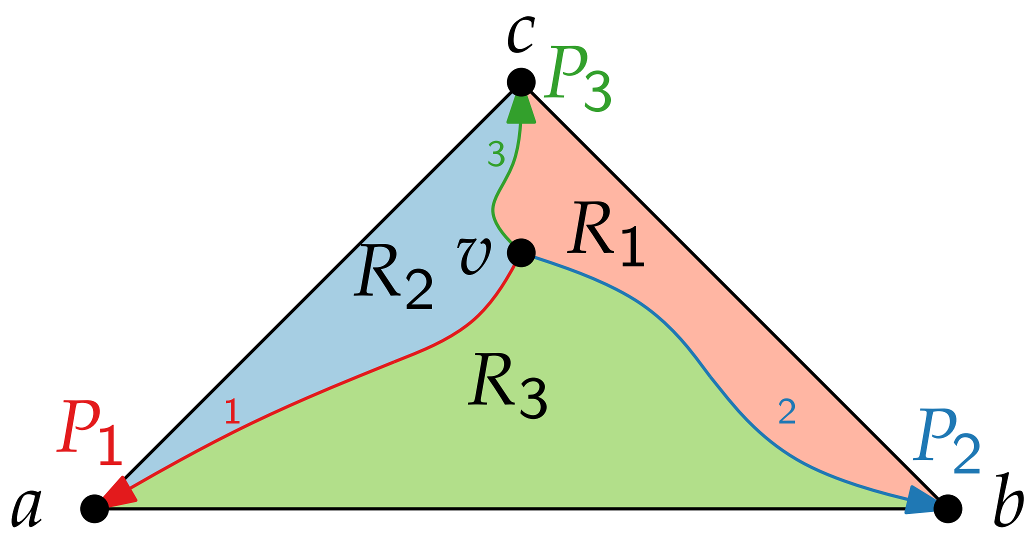
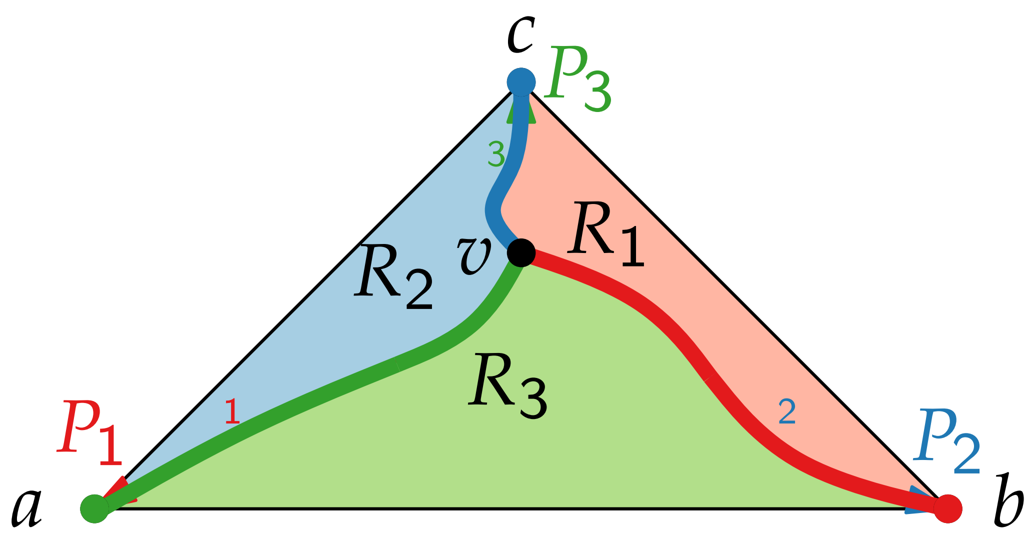


New barycentric coordinates

- Set $v'_i = |V(R_i(v))| - |P_{i-1}(v)|$



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- Set $v'_i = |V(R_i(v))| - |P_{i-1}(v)|$
- Additionally, for outer vertices set
  - $a'_1 = n - 2$
  - $a'_2 = 1$
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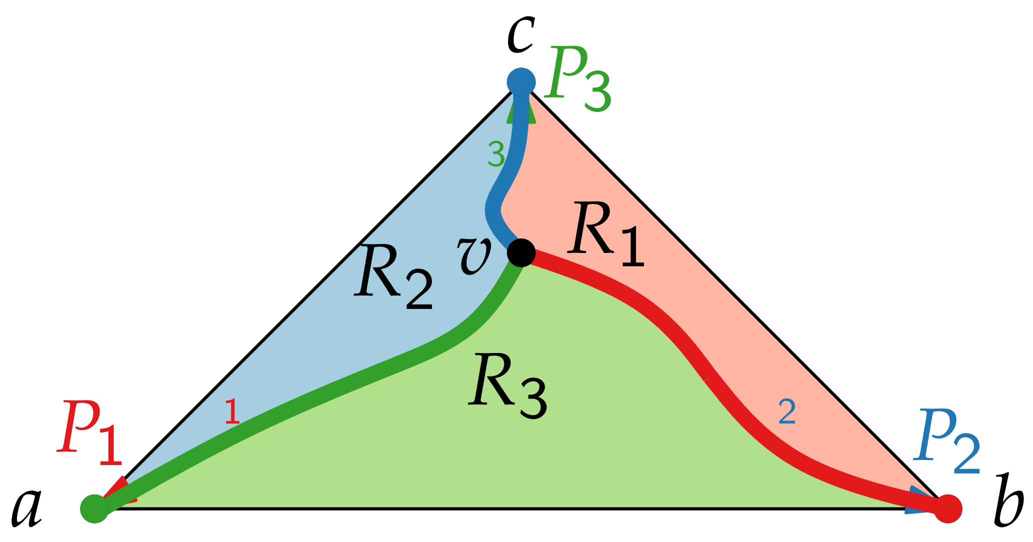
  and analogously for b' and c'



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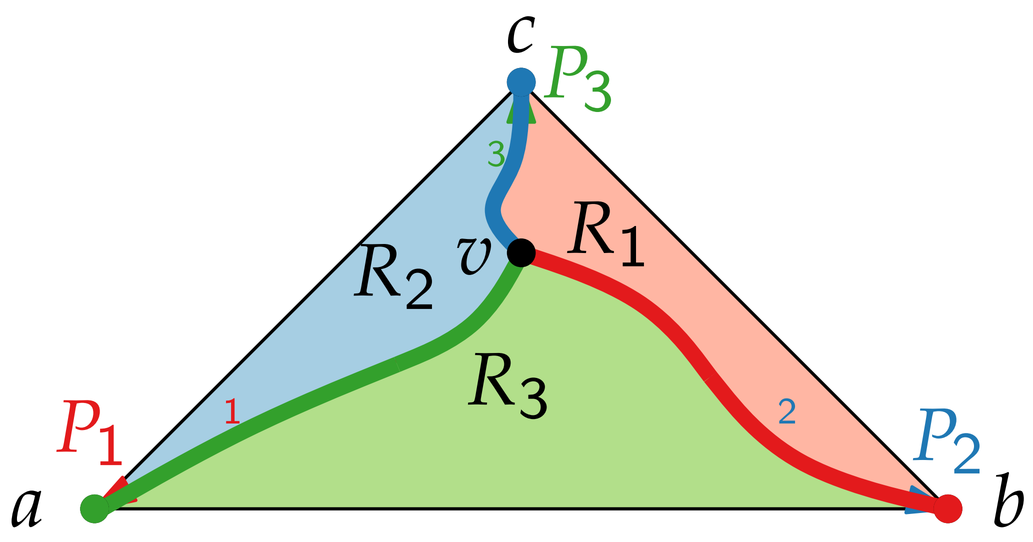

Lemma.
For inner vertices $u \neq v$ it holds that

$$u \in R_i(v) \Rightarrow (u'_i, u'_{i+1}) <_{\text{lex}} (v'_i, v'_{i+1})$$

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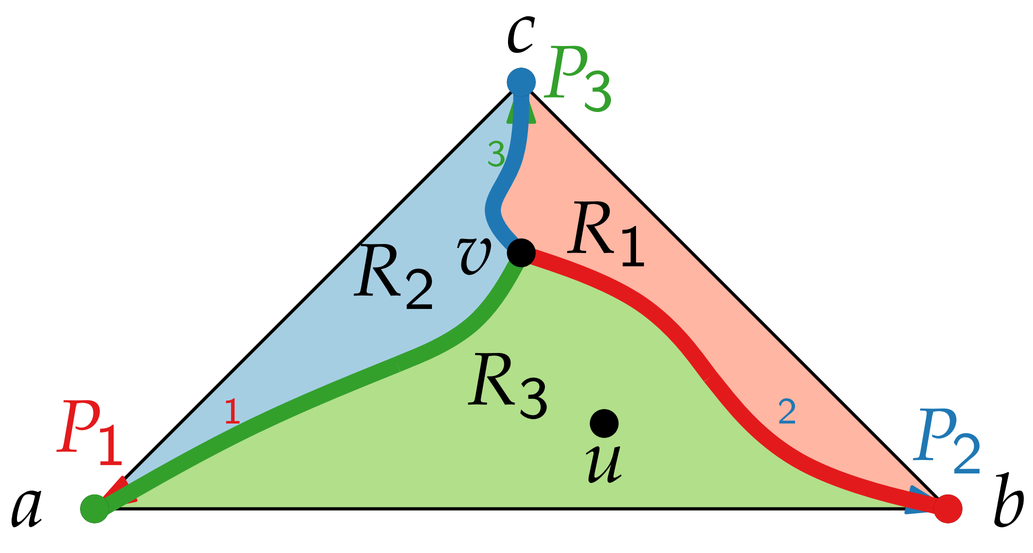

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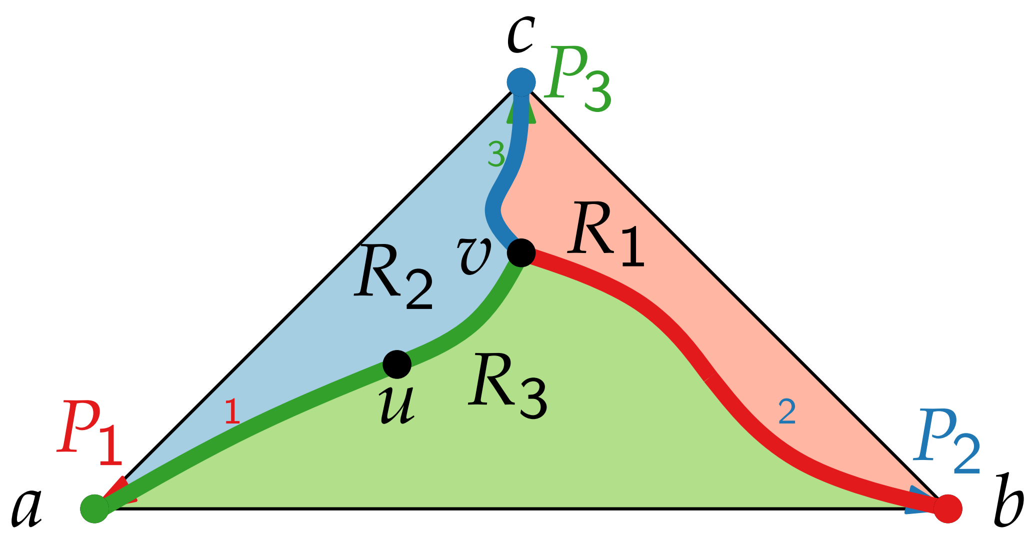

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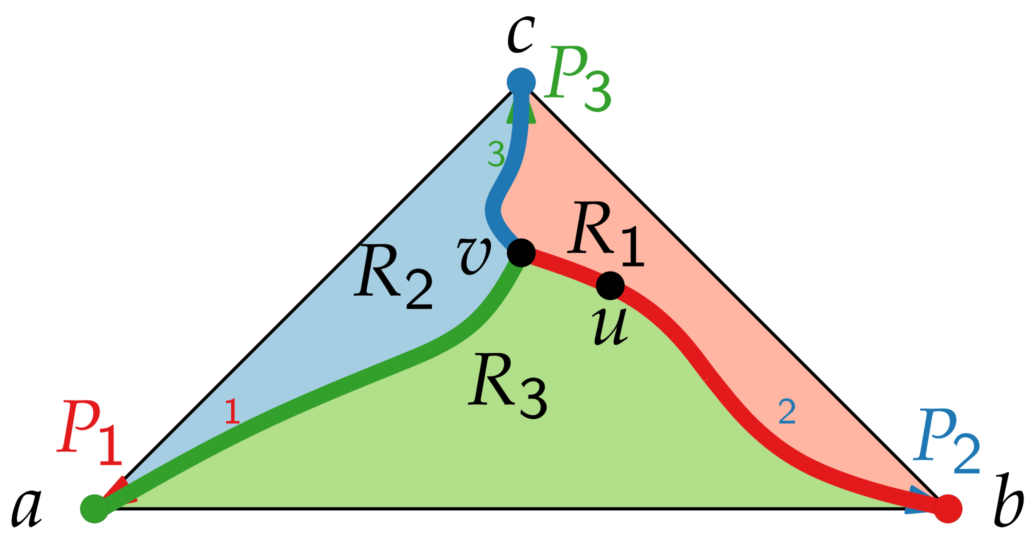

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$$f : v \mapsto \frac{1}{n-1}(v'_1, v'_2, v'_3)$$

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- To calculate all the coordinates, a constant number of tree traversals are enough.

Calculations

Compute:

- $p_i(v) = |P_i(v)|$ vertices on i-path from v
- $t_i(v) = |T_i(v)|$ vertices on i-subtree rooted at v
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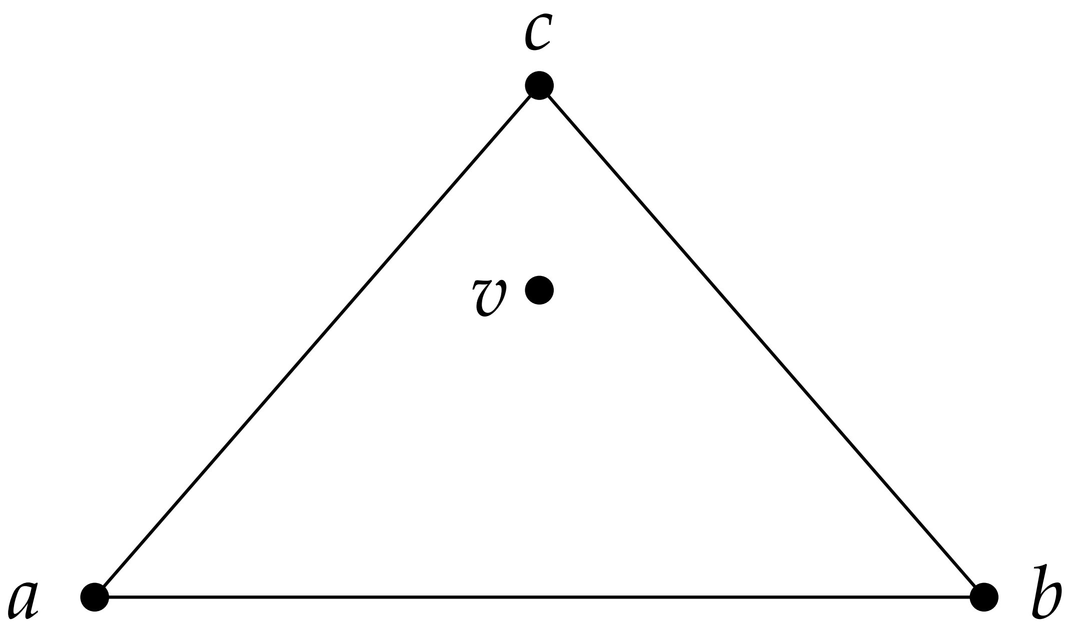
- $v'_i = |V(R_i(v))| - |P_{i-1}(v)|$ i-coordinate of v



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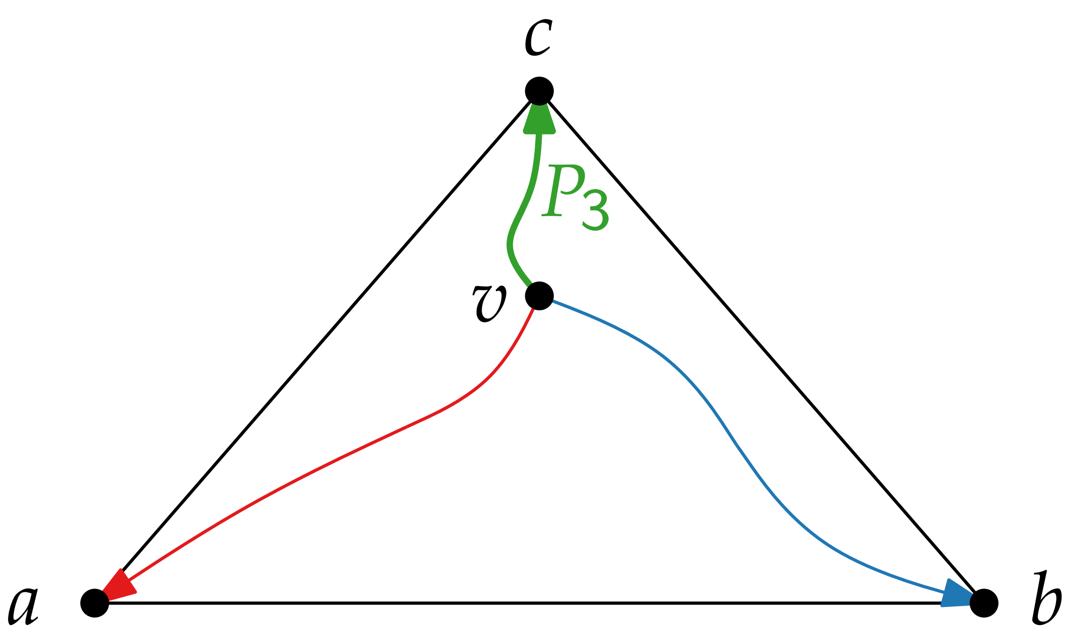

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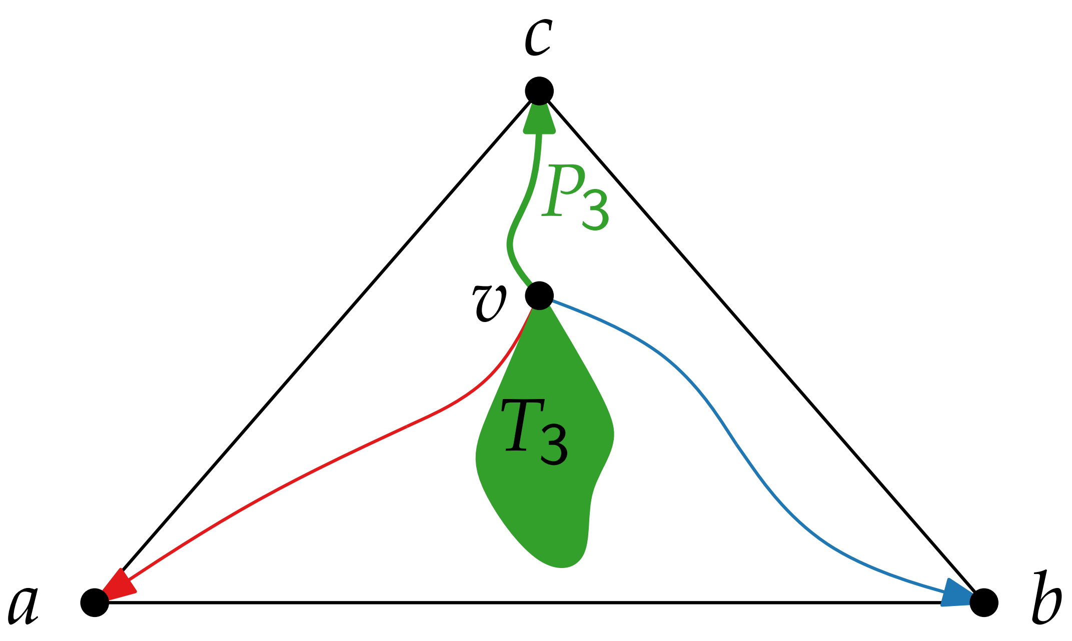

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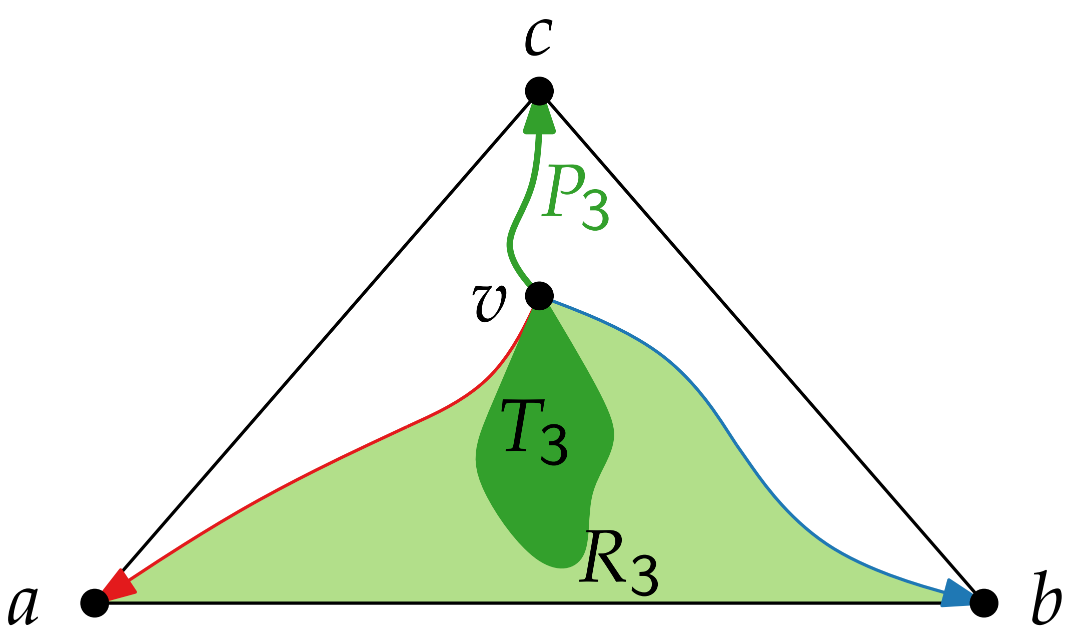

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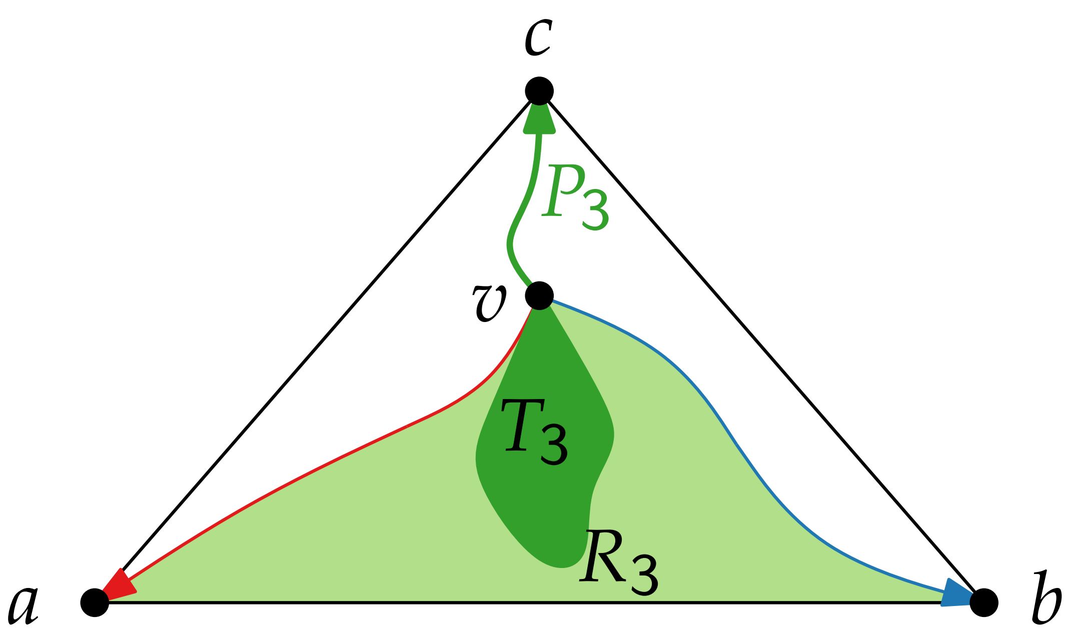

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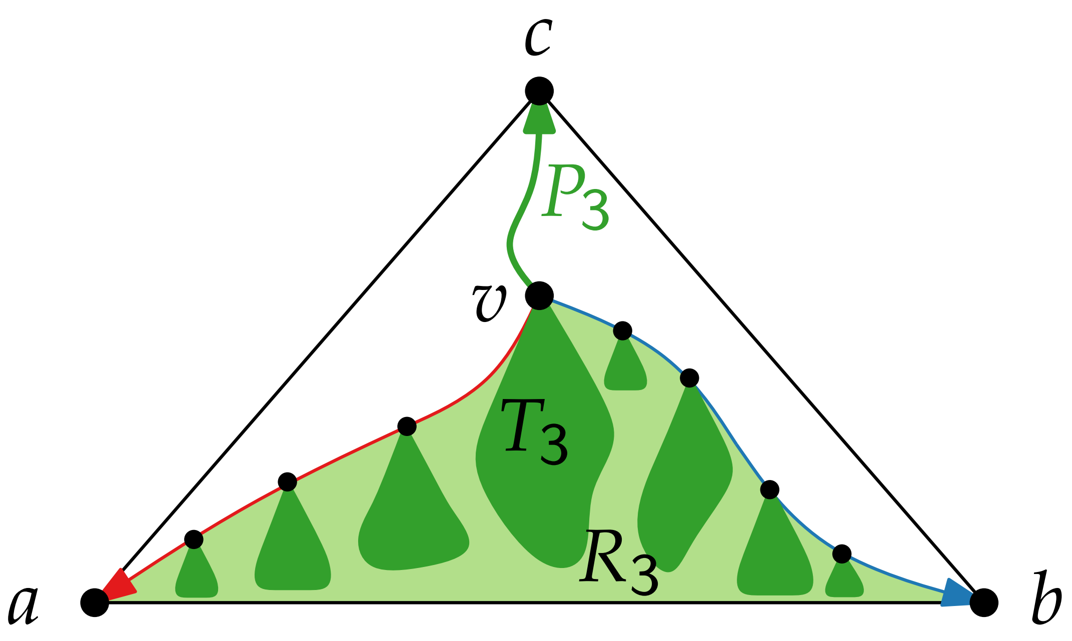

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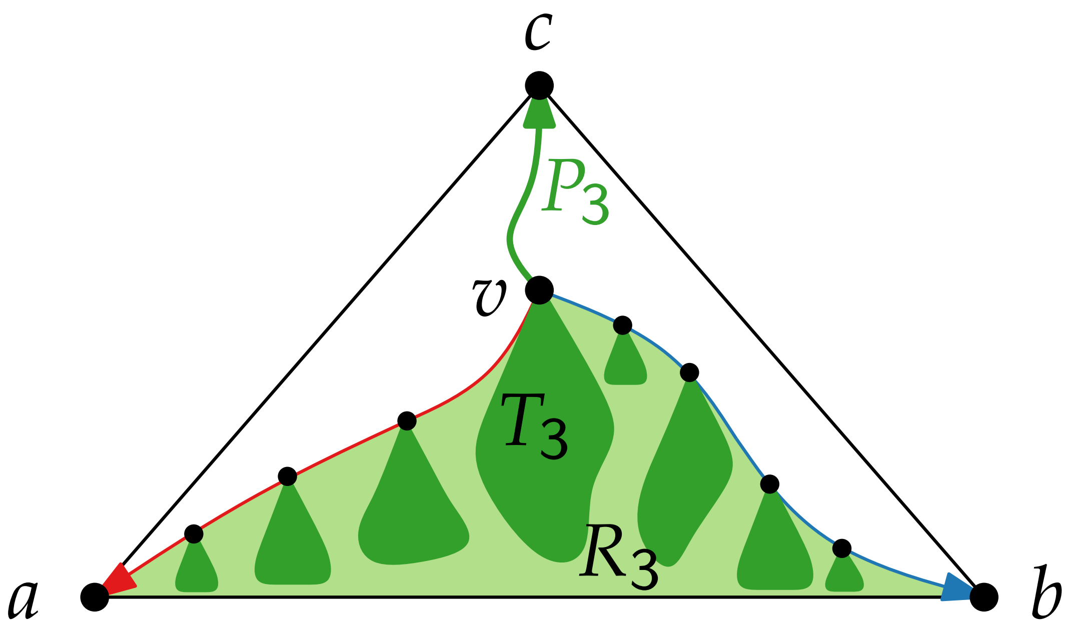
- $p_i(v)$ preorder
- $t_i(v)$ postorder
- $pt_i^j(v) = \sum_{u \in P_j(v)} t_i(u)$ preorder
- $r_i(v) = pt_i^{i-1}(v) + pt_i^{i+1}(v) - t_i(v)$
- $v'_i = r_i(v) - p_{i-1}(v)$



Literature

- [Sch90] Schnyder "Embedding planar graphs on the grid" 1990 – original paper on Schnyder realiser method