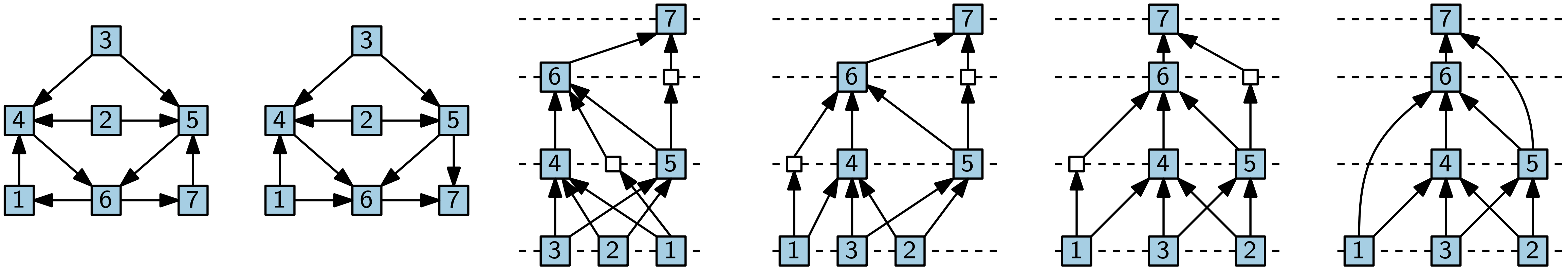
Visualisation of graphs

Hierarchical layouts

Sugiyama framework

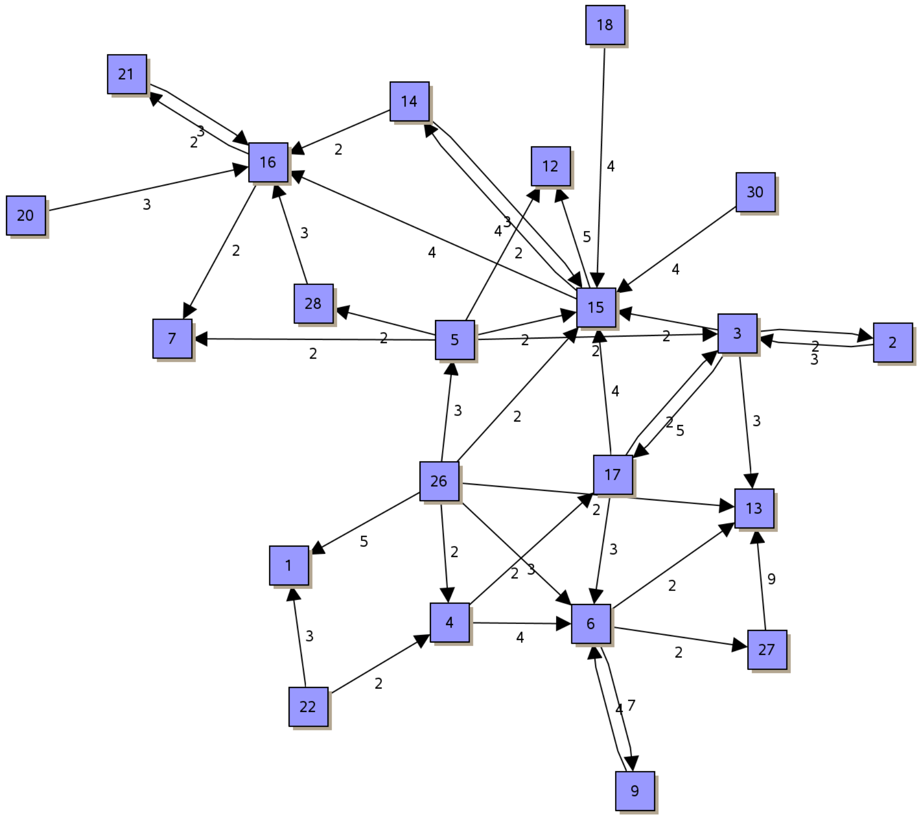
Antonios Symvonis · Chrysanthi Raftopoulou

Fall semester 2020

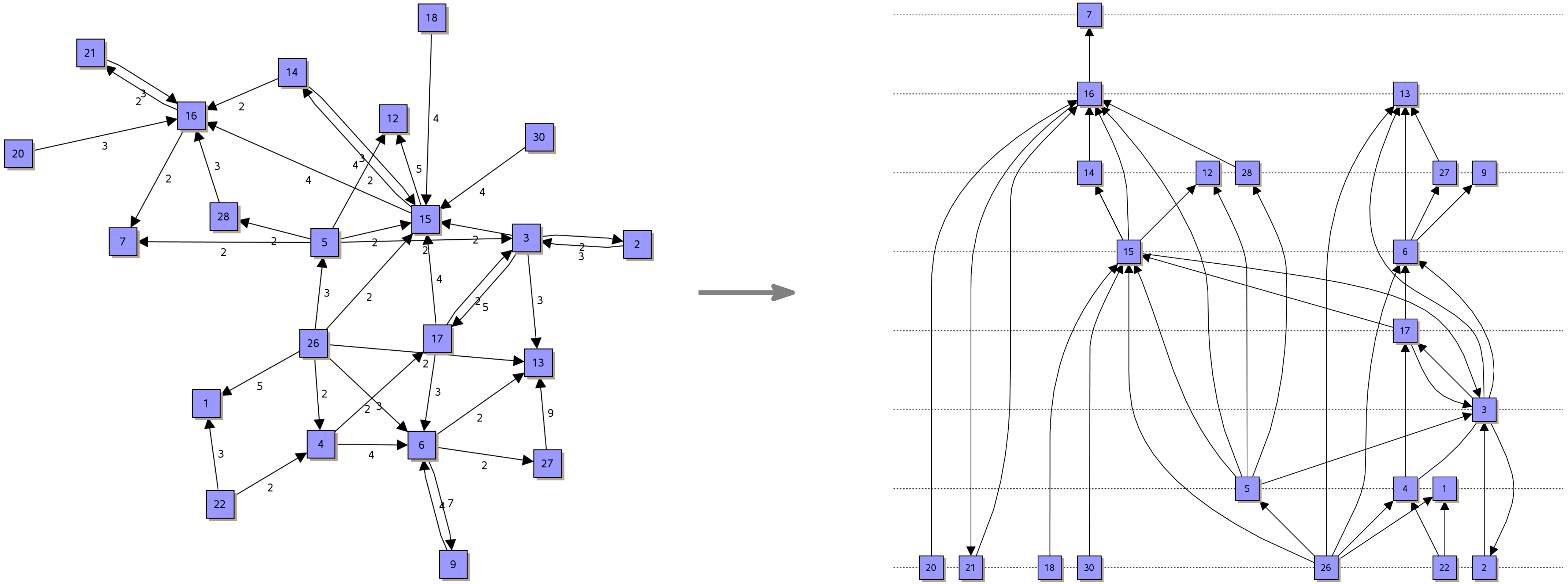


The original slides of this presentation were created by researchers at Karlsruhe Institute of Technology (KIT), TU Wien, U Wuerzburg, U Konstanz, ...
The original presentation was modified/updated by A. Symvonis and C. Raftopoulou

Hierarchical drawings – motivation



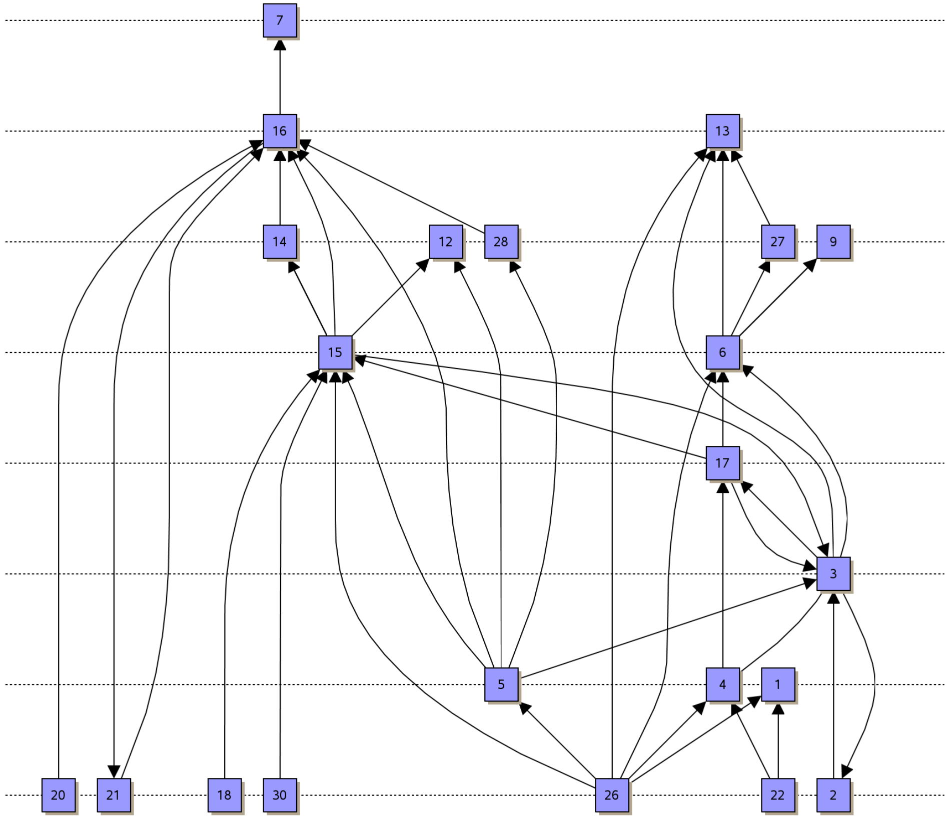
Hierarchical drawings – motivation



Hierarchical drawing

Problem statement.

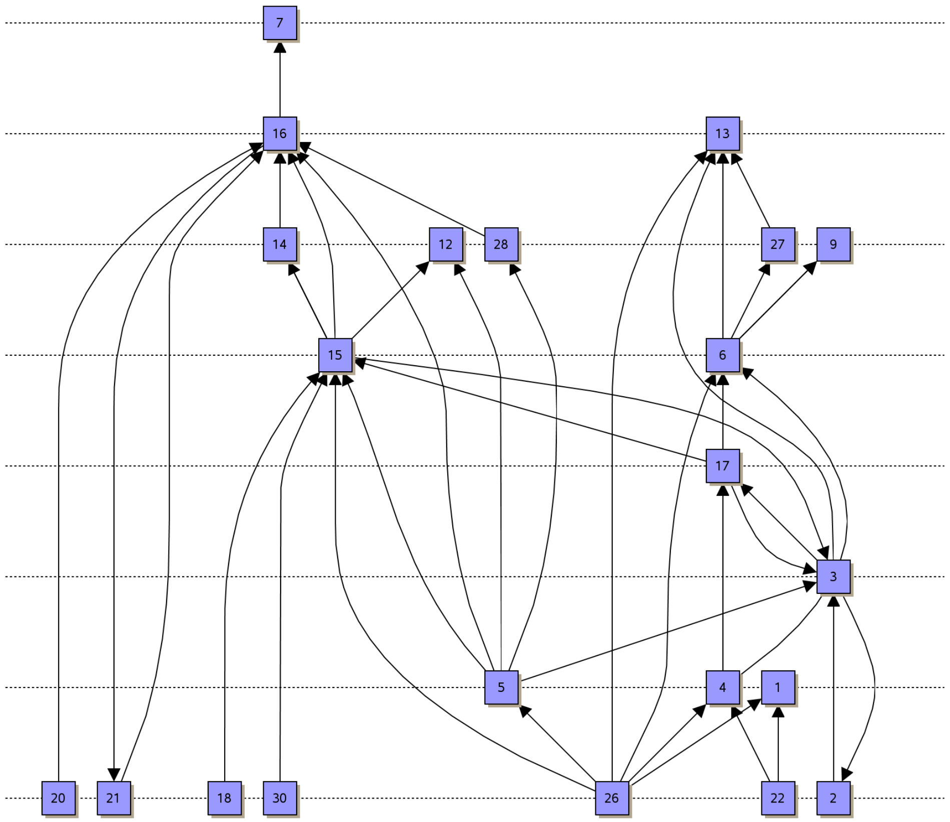
- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G



Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G



Desireable properties.

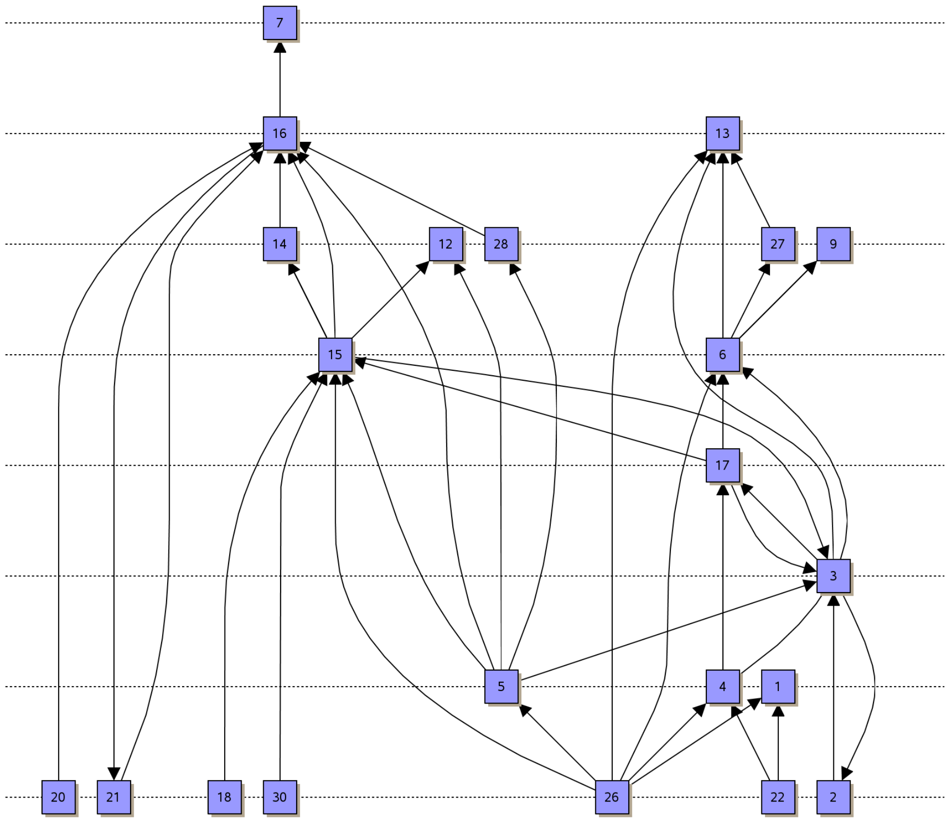
Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desireable properties.

- vertices occur on (few) horizontal lines



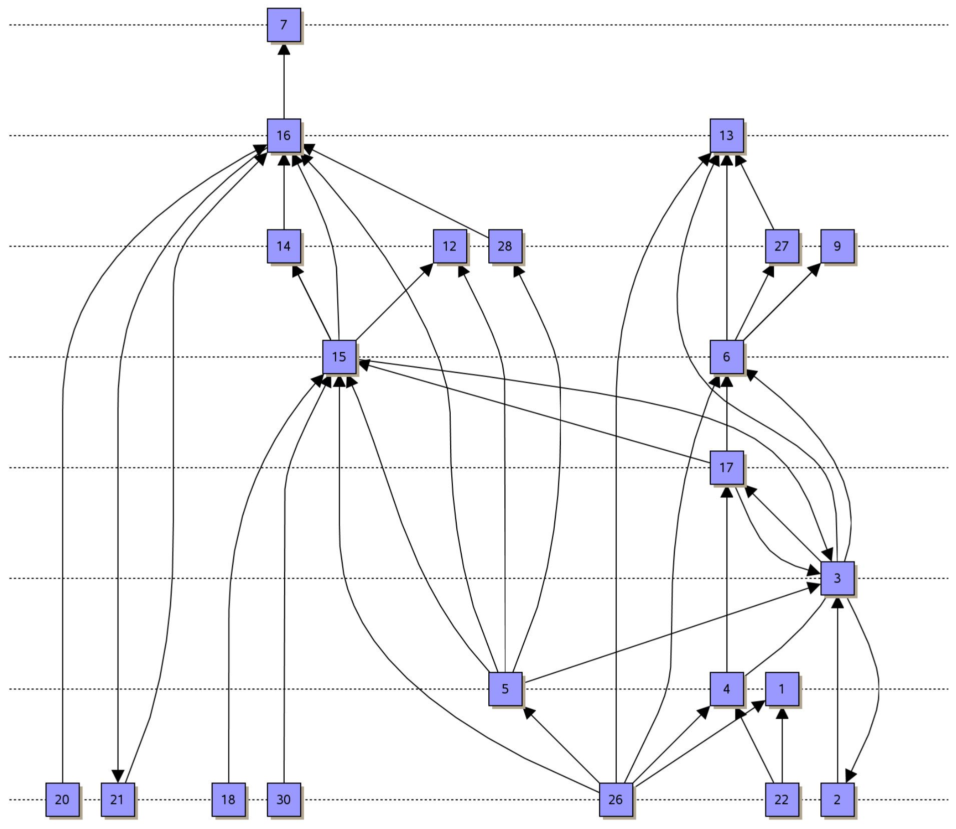
Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desireable properties.

- vertices occur on (few) horizontal lines
- edges directed upwards



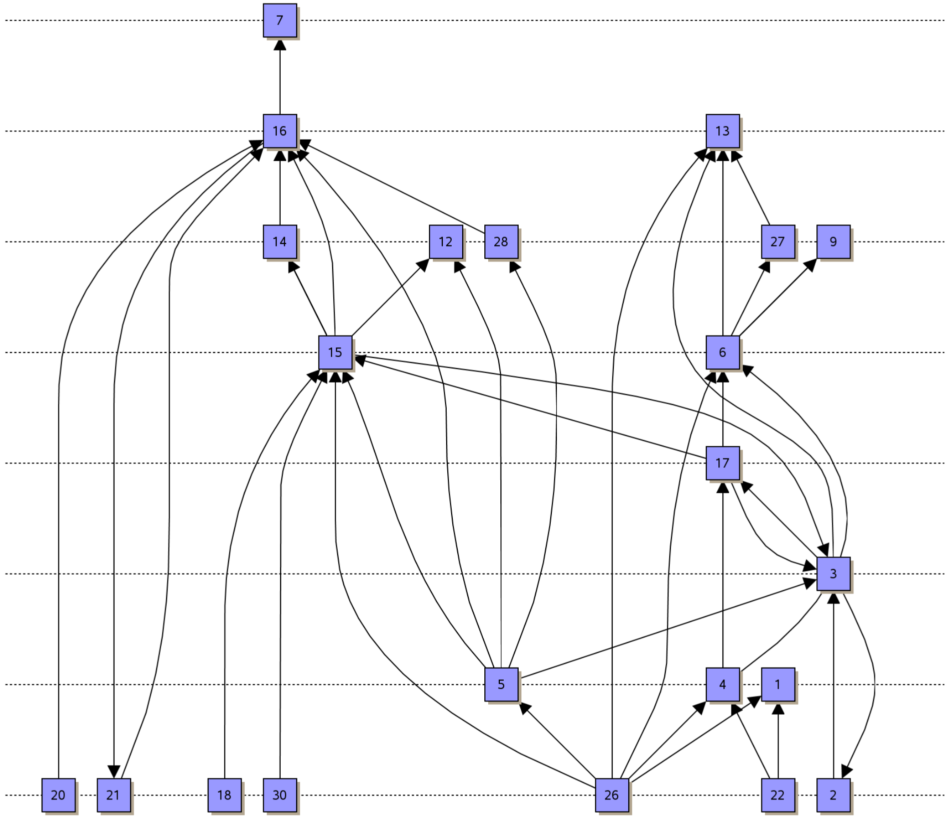
Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that “closely” reproduces the hierarchical properties of G

Desireable properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized



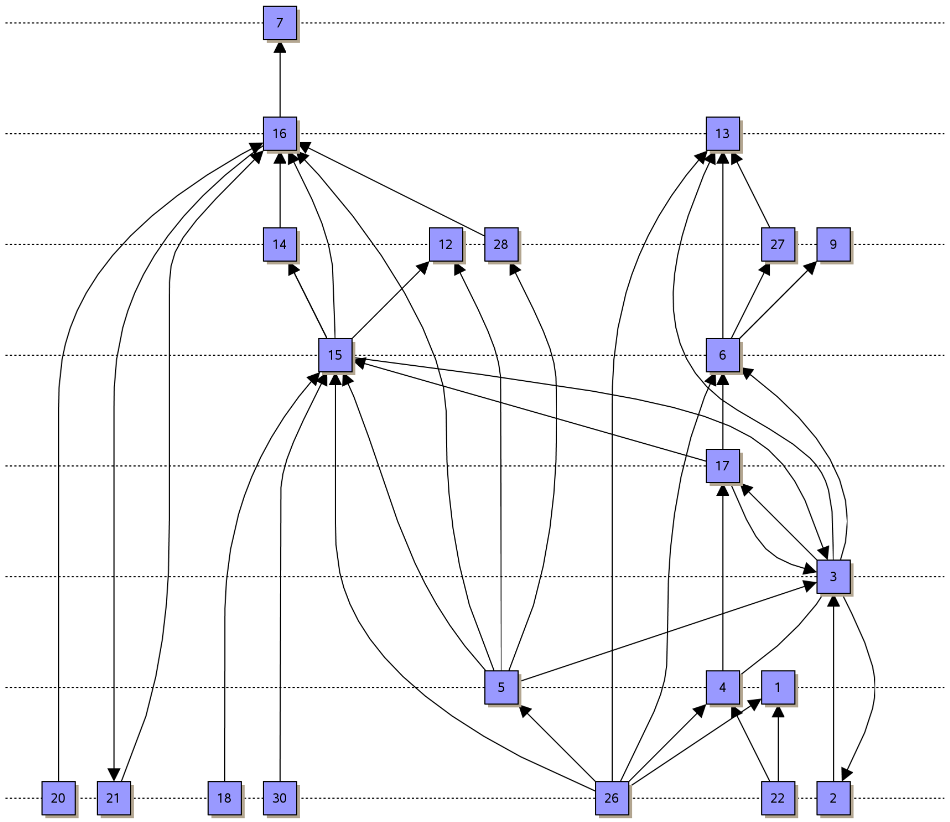
Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desireable properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges upward, straight, and short as possible



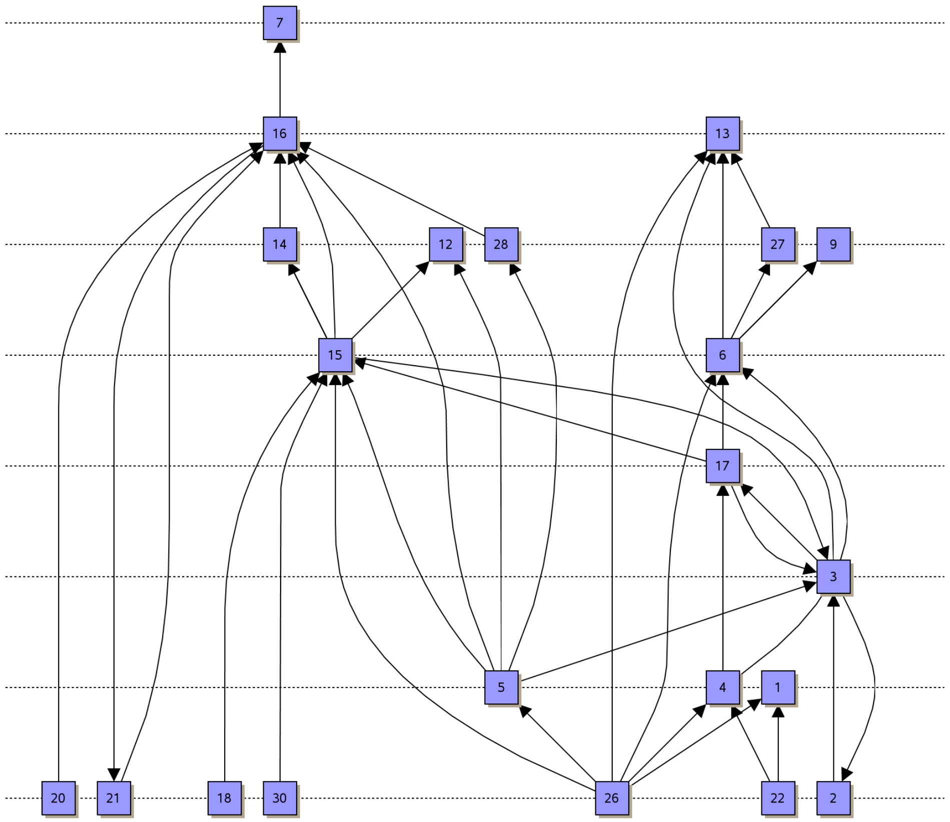
Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desireable properties.

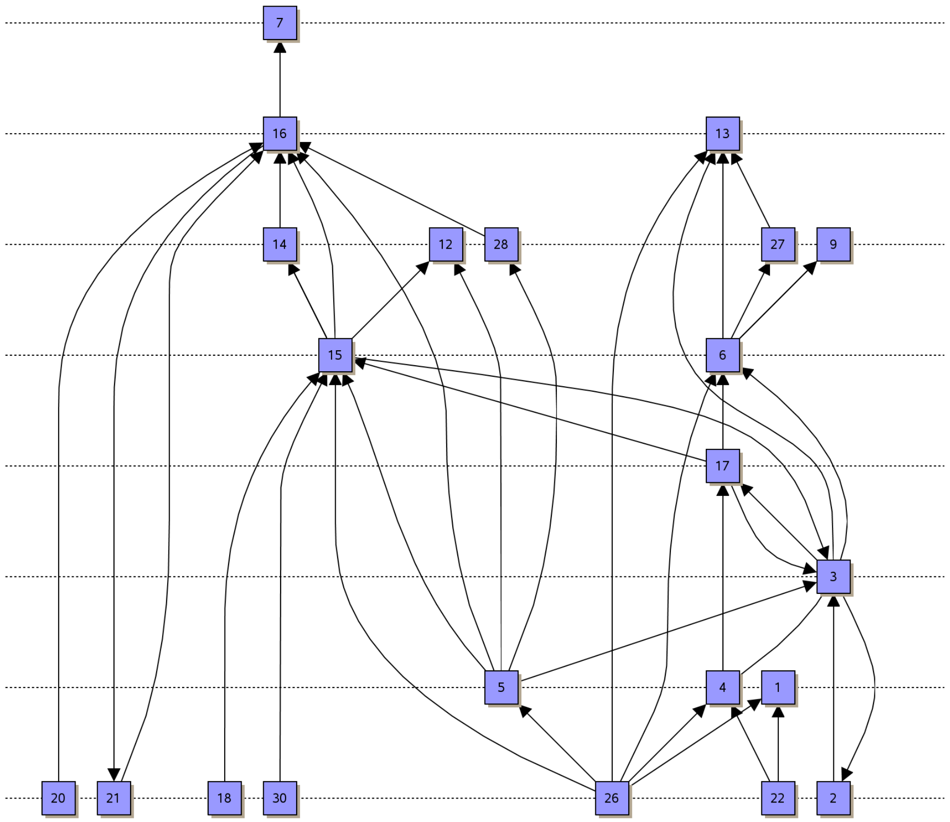
- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges upward, straight, and short as possible
- vertices evenly spaced



Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G



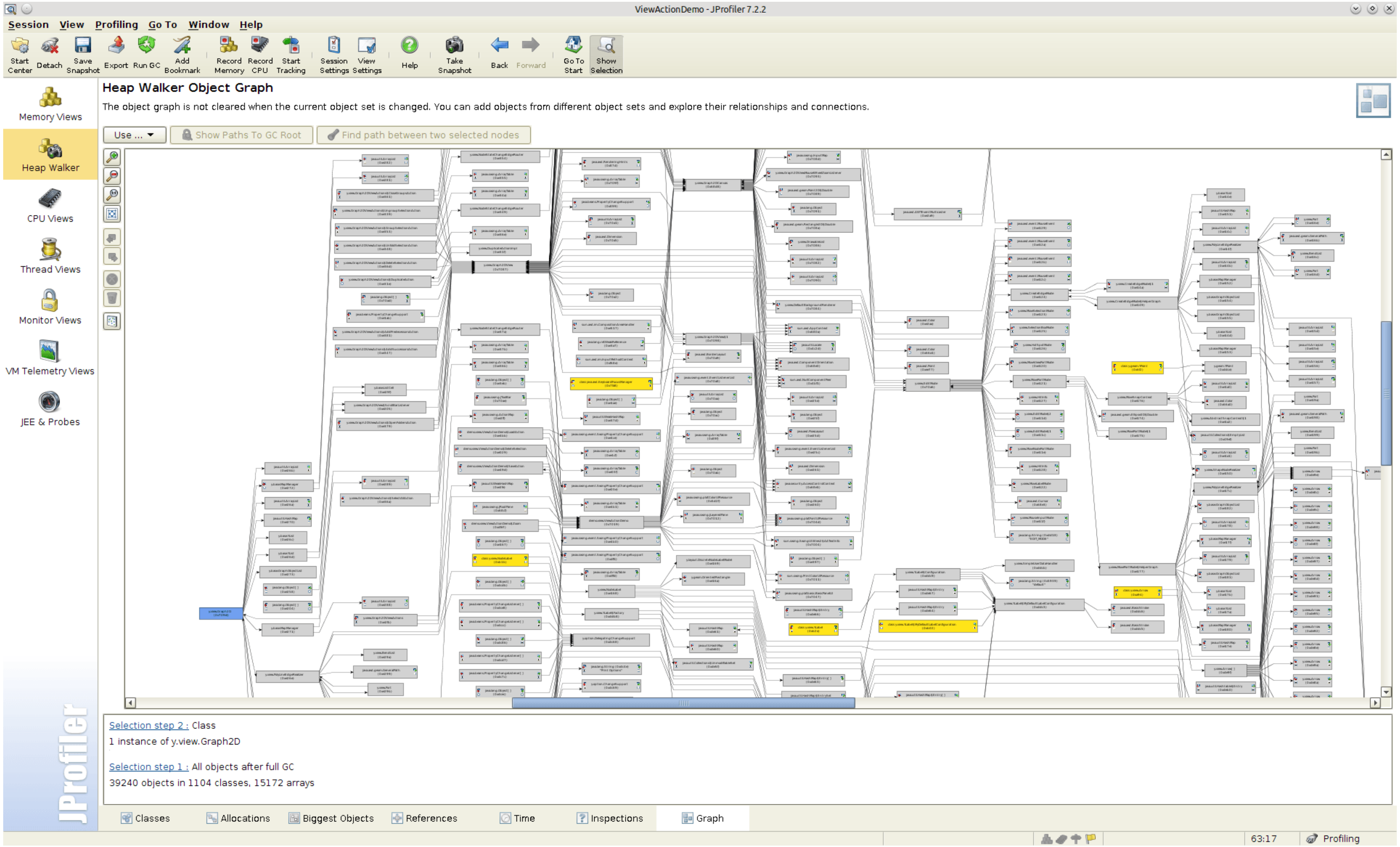
Desireable properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges upward, straight, and short as possible
- vertices evenly spaced

Criteria can be contradictory!

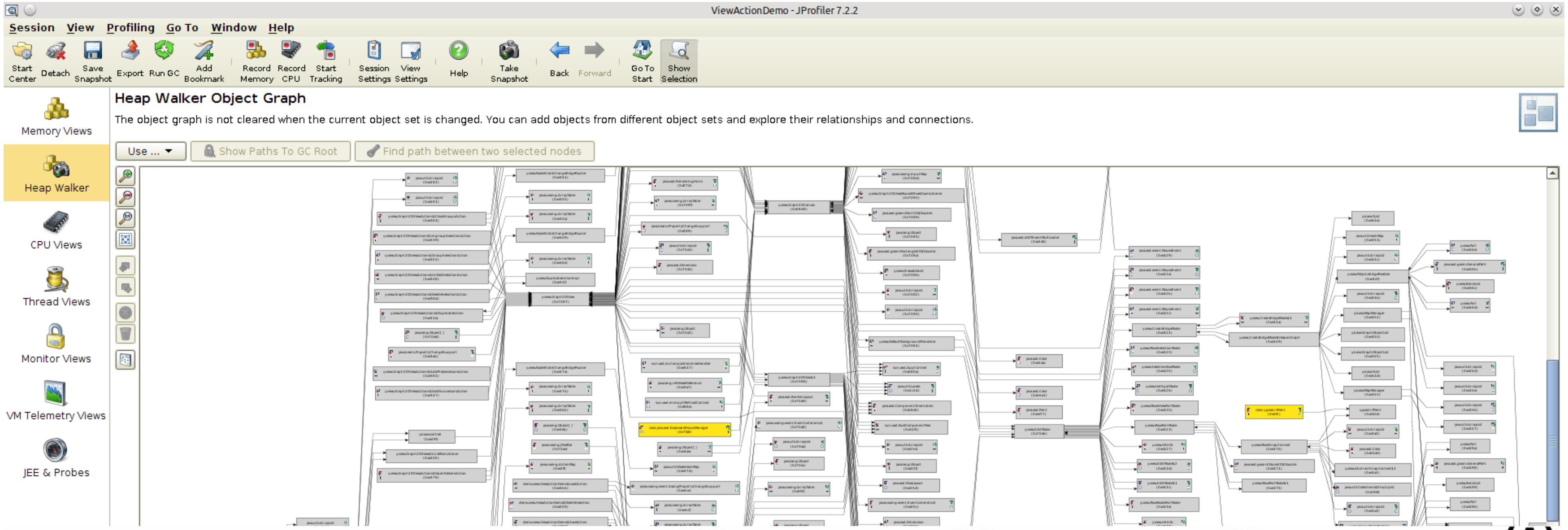
Hierarchical drawing – applications

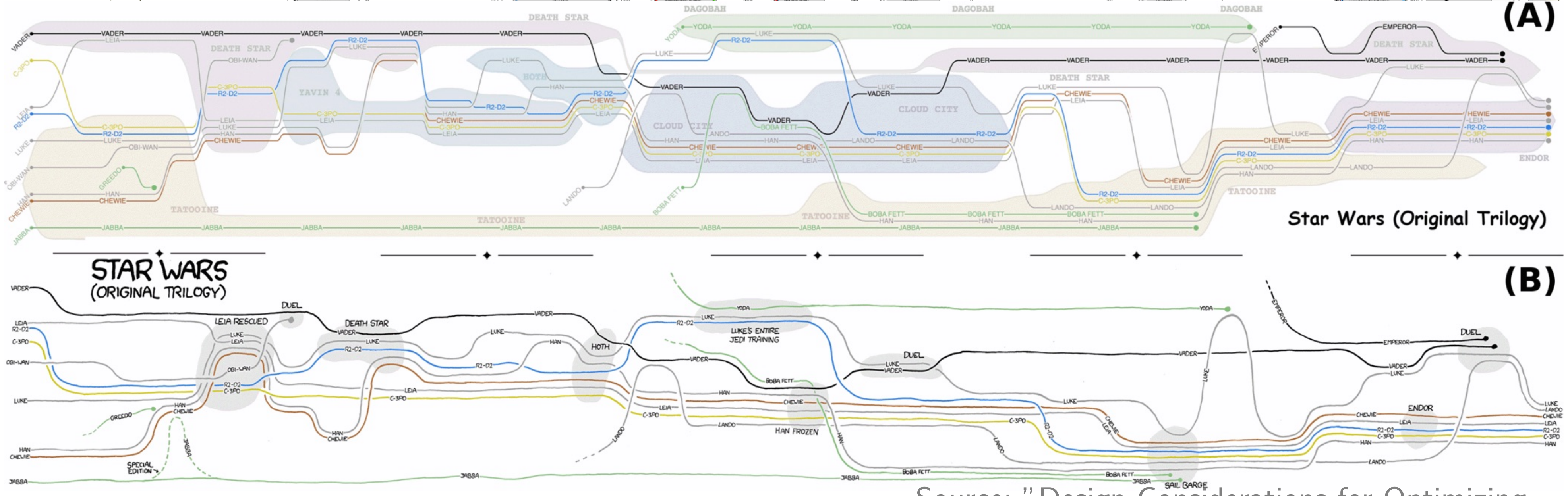
yEd Gallery: Java profiler JProfiler using yFiles



Hierarchical drawing – applications

yEd Gallery: Java profiler JProfiler using yFiles

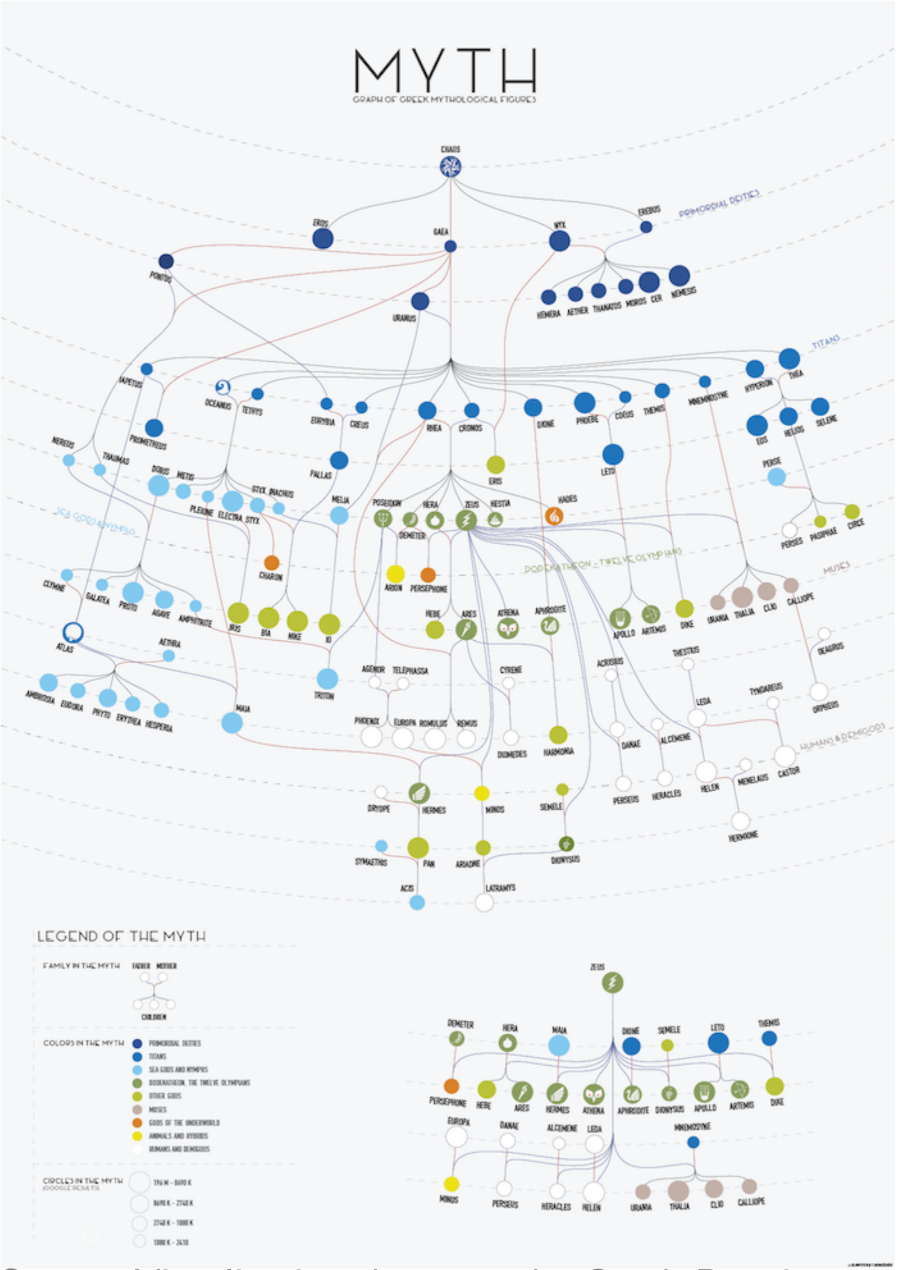


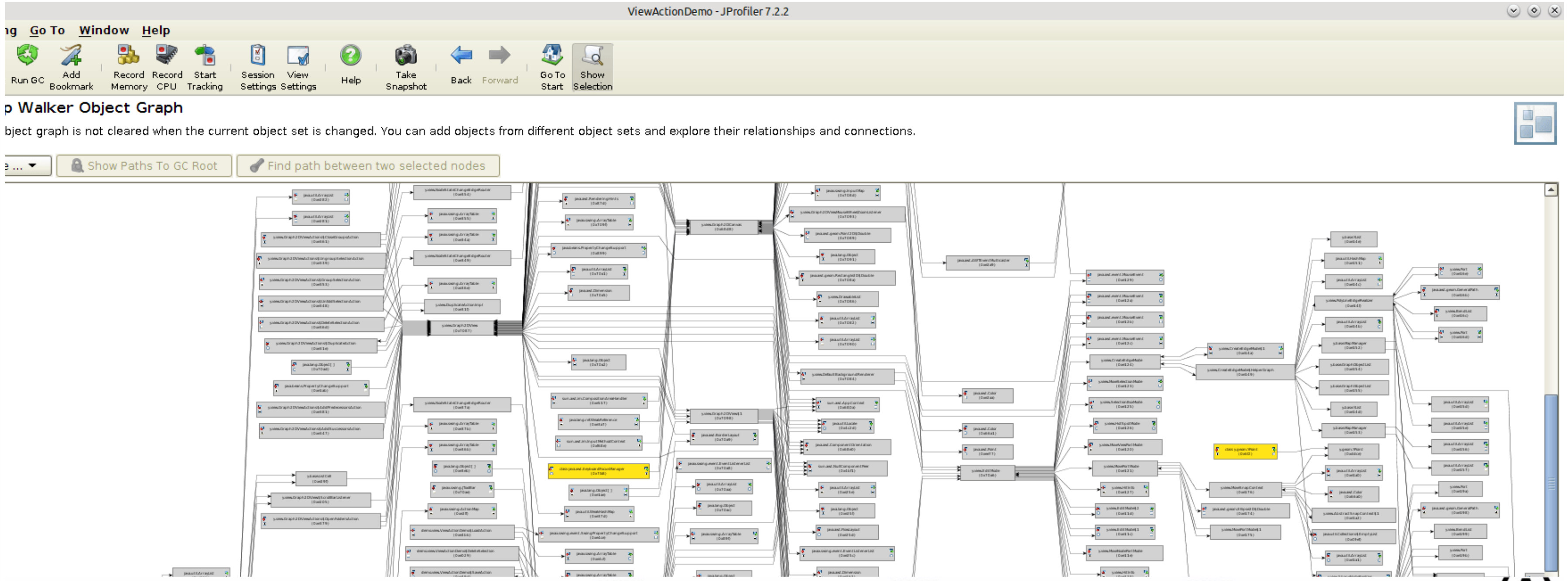


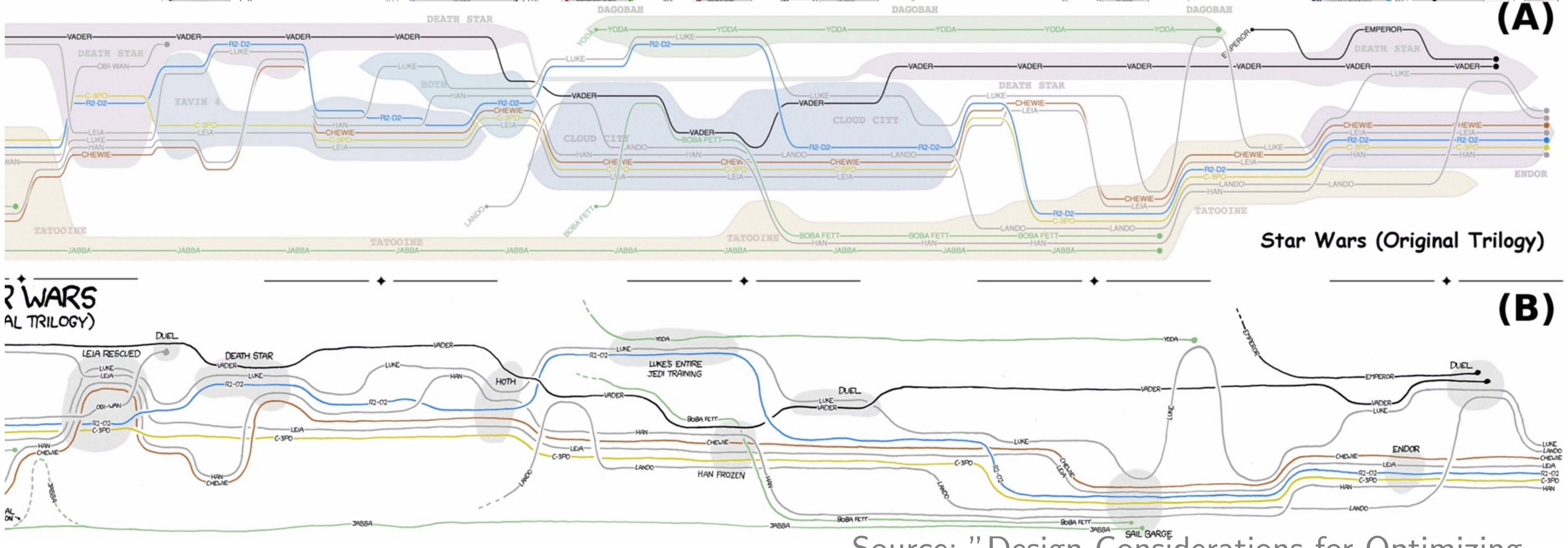
Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

Hierarchical drawing – applications

yEd Gallery: Java profiler JProfiler using yFiles







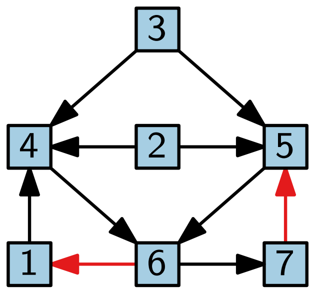
Source: Visualization that won the Graph Drawing contest 2016. Klawitter & Mchedlidze

Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]

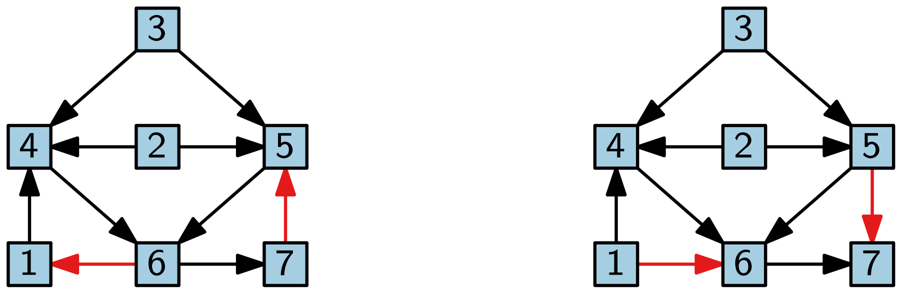
Input



Classical approach – Sugiyama framework

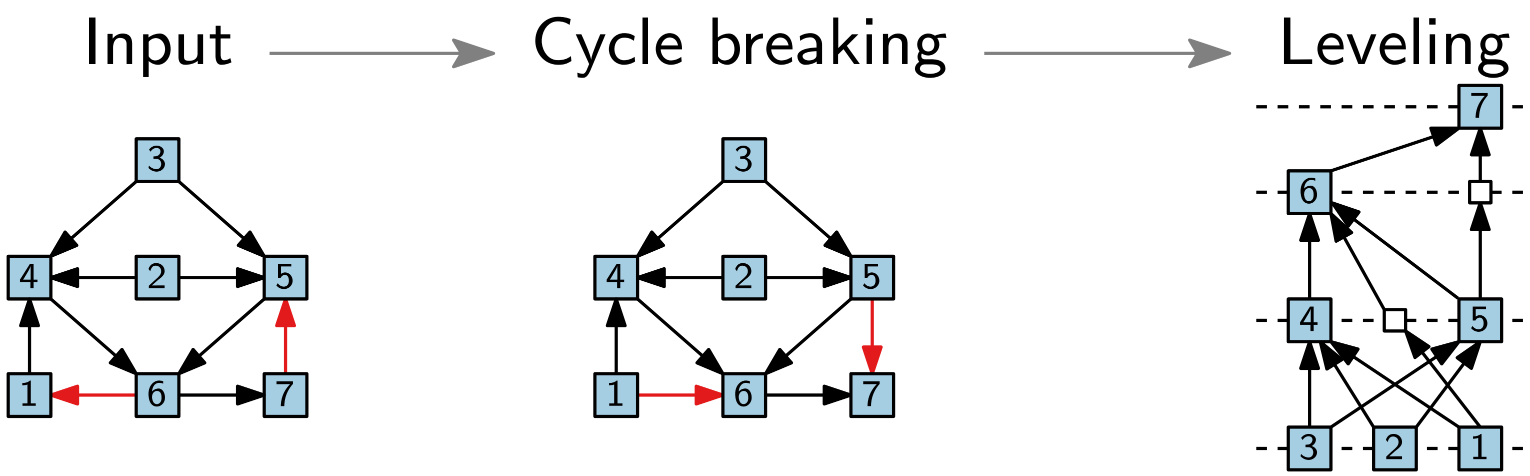
[Sugiyama, Tagawa, Toda '81]

Input → Cycle breaking



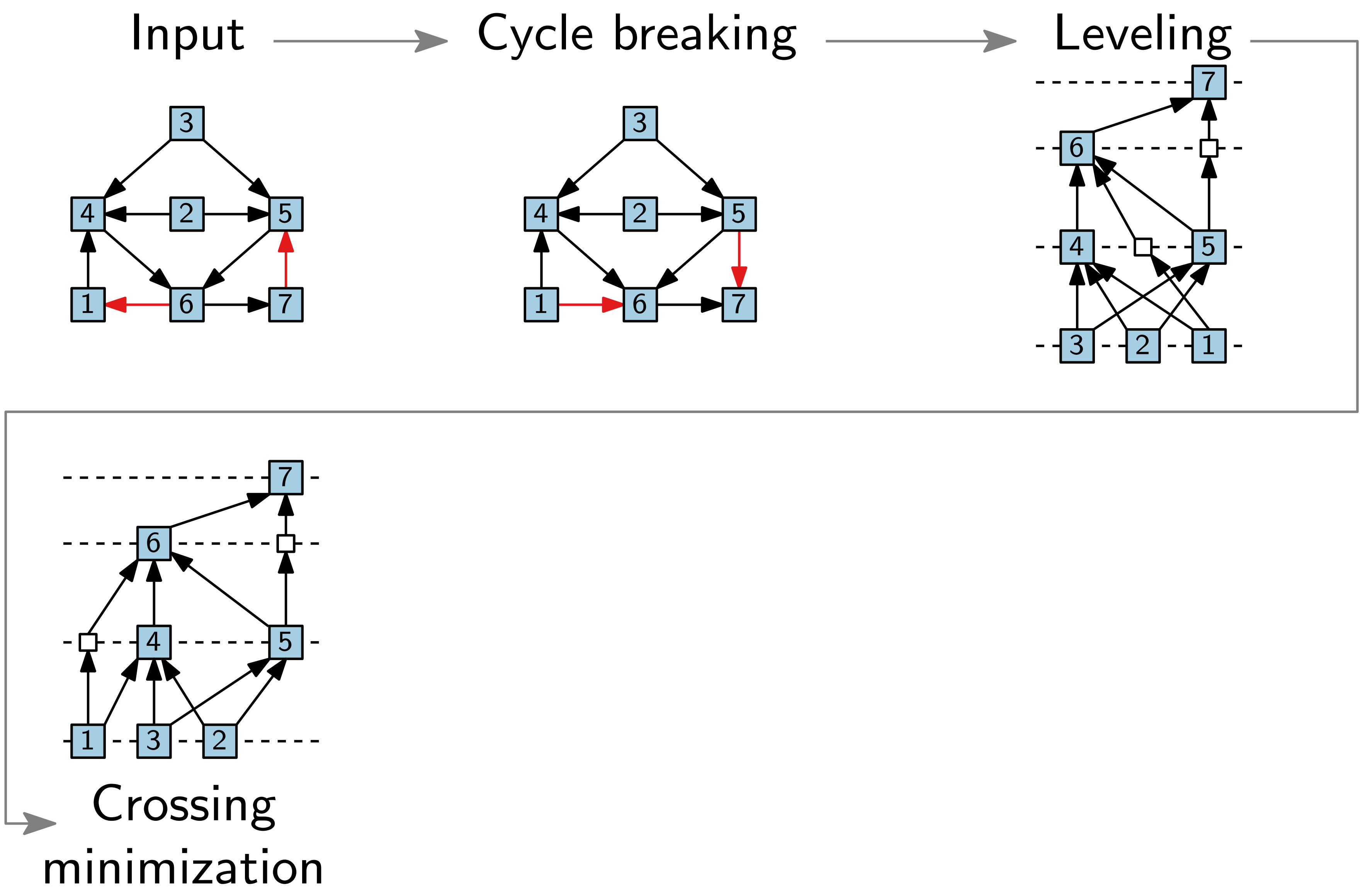
Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]



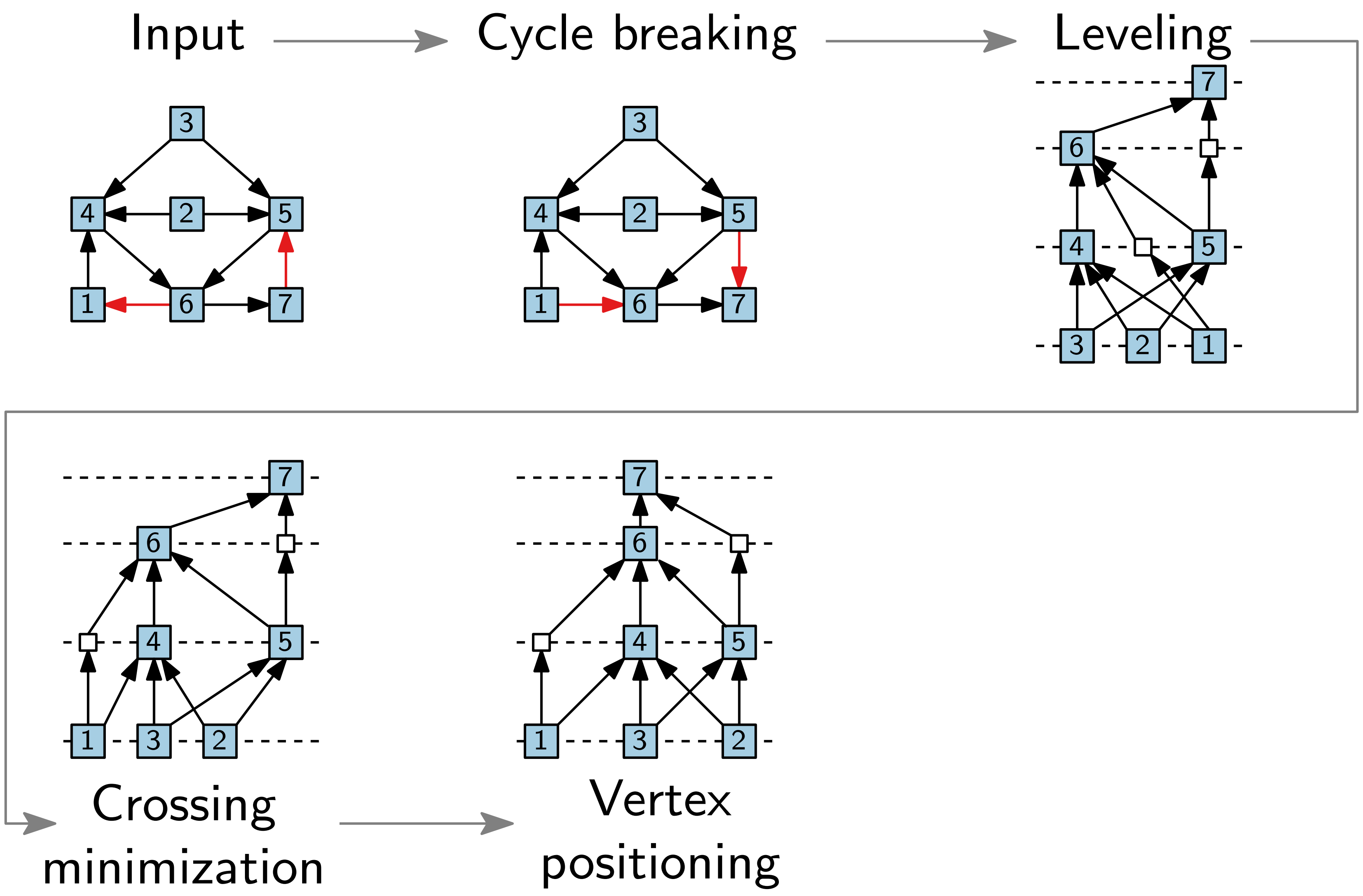
Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda '81]



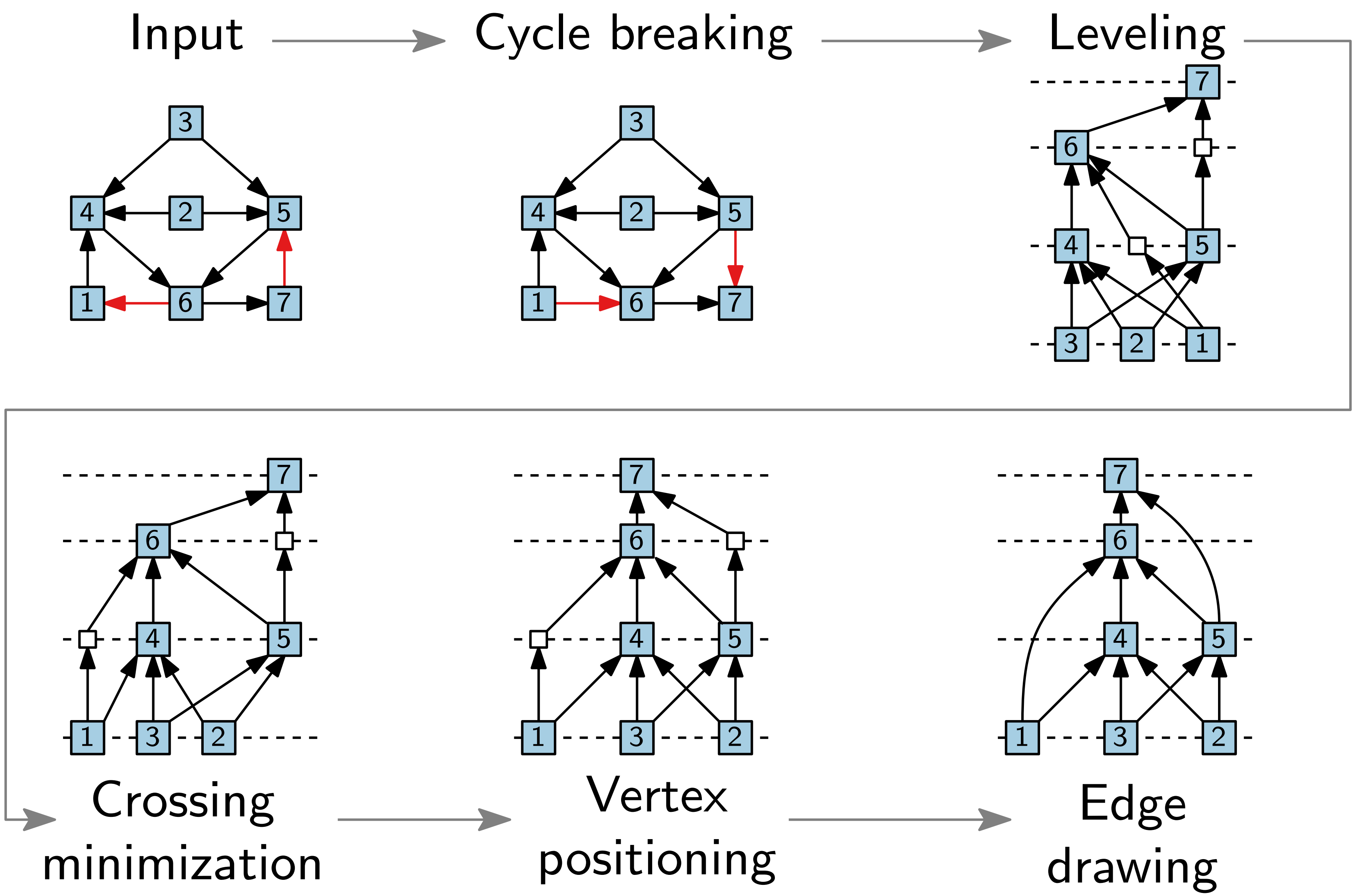
Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]

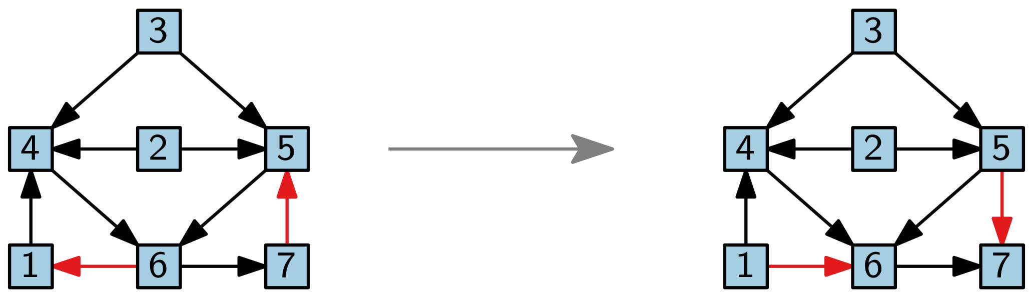


Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]



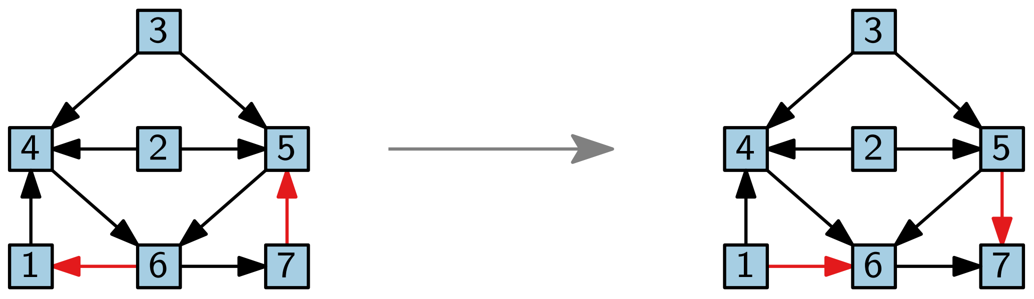
Step 1: Cycle breaking



Approach.

- Find minimum set $E^\star$ of edges which are not upwards.
- Remove $E^\star$ and insert reversed edges.

Step 1: Cycle breaking



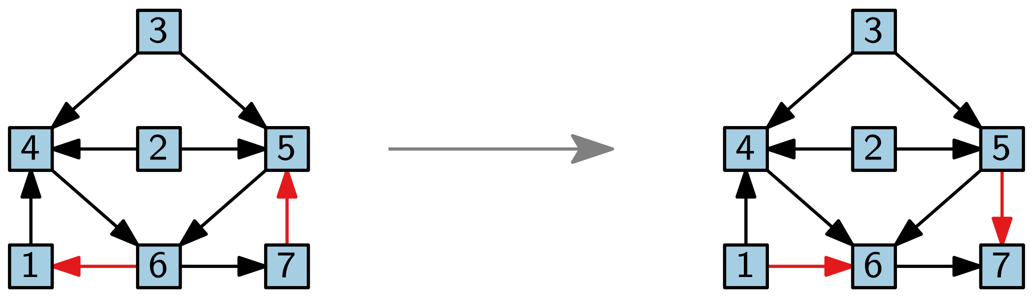
Approach.

- Find minimum set $E^\star$ of edges which are not upwards.
- Remove $E^\star$ and insert reversed edges.

Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

Step 1: Cycle breaking



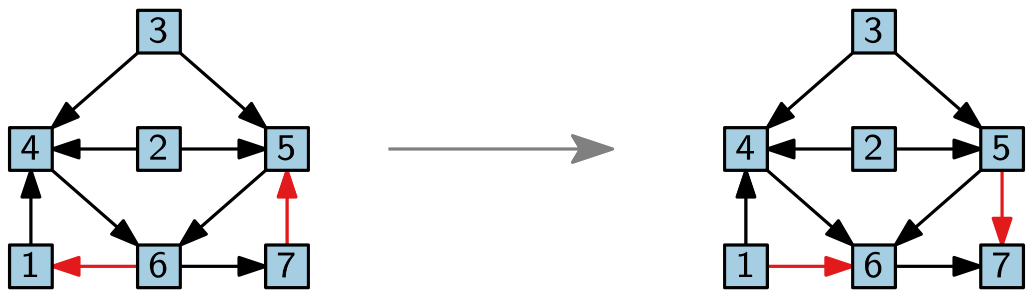
Approach.

- Find minimum set $E^\star$ of edges which are not upwards.
- Remove $E^\star$ and insert reversed edges.

Problem MINIMUM FEEDBACK ~~ARC~~ SET **(F~~A~~S).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that ~~$G - E^\star$~~ acyclic
  $G - E^\star + E_r^\star$

Step 1: Cycle breaking



Approach.

- Find minimum set $E^\star$ of edges which are not upwards.
- Remove $E^\star$ and insert reversed edges.

Problem MINIMUM FEEDBACK ~~ARC~~ SET **(F~~A~~S).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that ~~$G - E^\star$~~ $G - E^\star + E_r^\star$ acyclic

. . . NP-hard :-(

Step 1: Cycle breaking

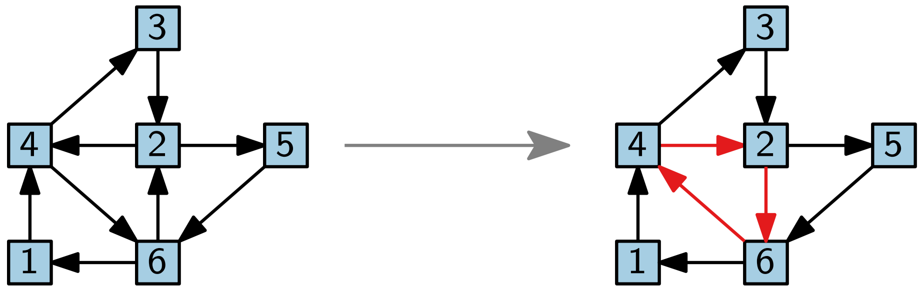
Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

Step 1: Cycle breaking

Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

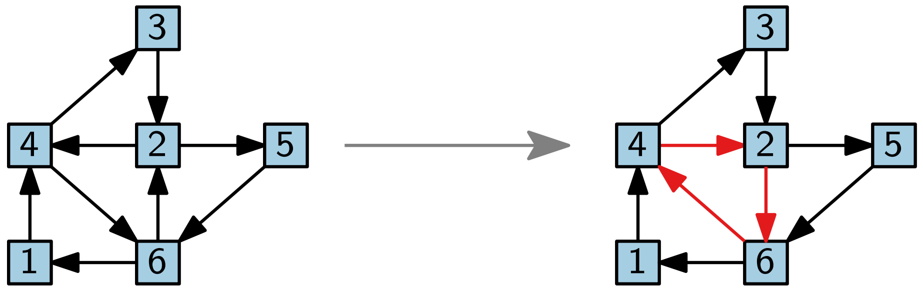


Step 1: Cycle breaking

Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

$G - E^\star + E^\star_r$ not acyclic

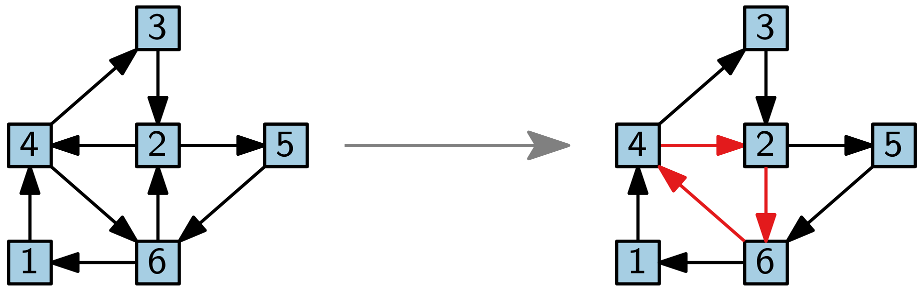


Step 1: Cycle breaking

Problem Minimum Feedback Arc Set **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

$G - E^\star + E^\star_r$ not acyclic



Problem Minimum Feedback Set **(FS).**

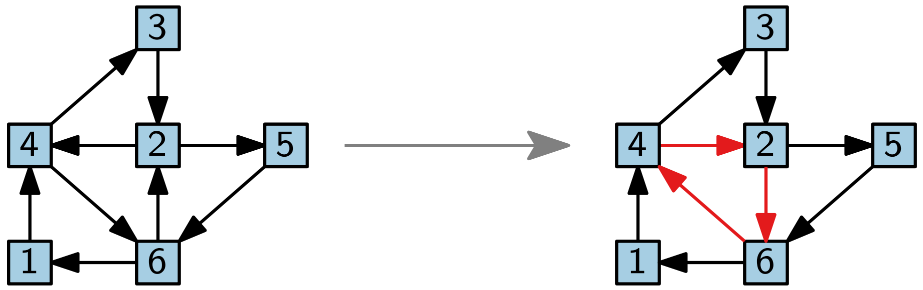
- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star + E^\star_r$ acyclic

Step 1: Cycle breaking

Problem Minimum Feedback Arc Set **(FAS).**

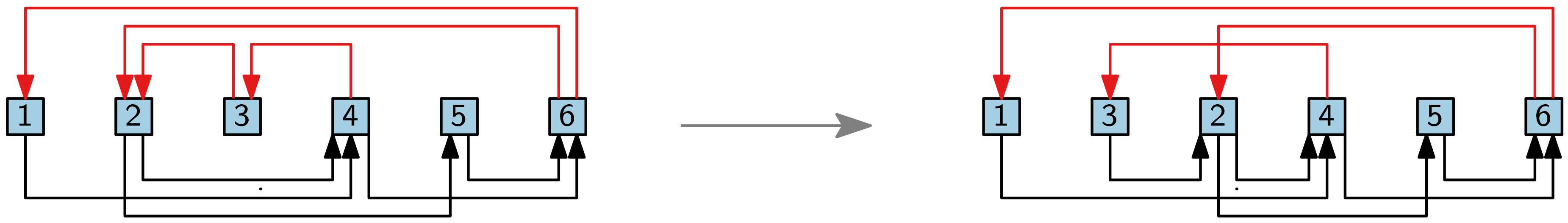
- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

$G - E^\star + E_r^\star$ not acyclic



Problem Minimum Feedback Set **(FS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star + E_r^\star$ acyclic



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$
foreach $v \in V$ **do**
  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  else
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  remove v and $N(v)$ from G.
return (V, E')

$$N^{\rightarrow}(v) := \{(v, u) | (v, u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u, v) | (u, v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

- $G' = (V, E')$ is a DAG

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

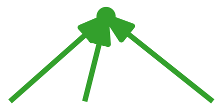
  remove v and $N(v)$ from G.

return (V, E')

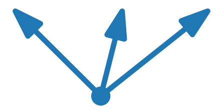
$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$

$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$



- $G' = (V, E')$ is a DAG



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

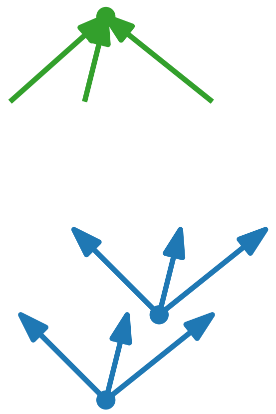
return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$

$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

- $G' = (V, E')$ is a DAG



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

else

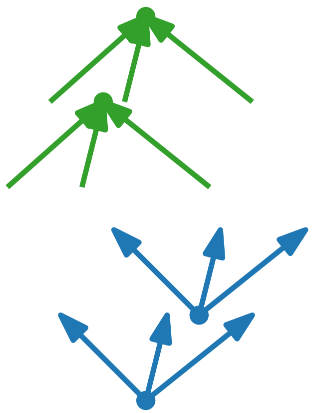
$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

- $G' = (V, E')$ is a DAG



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

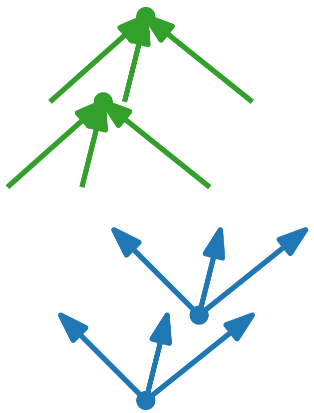
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v, u) | (v, u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u, v) | (u, v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

- $G' = (V, E')$ is a DAG
  - we create an order on V



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

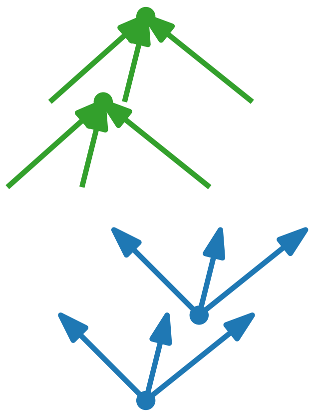
else

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$



- $G' = (V, E')$ is a DAG
  - we create an order on V
- $E \setminus E'$ is a feedback arc set

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

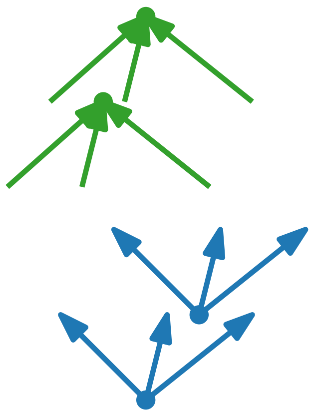
  remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$

$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$



- $G' = (V, E')$ is a DAG
  - we create an order on V
- $E \setminus E'$ is a feedback arc set

- Time:

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

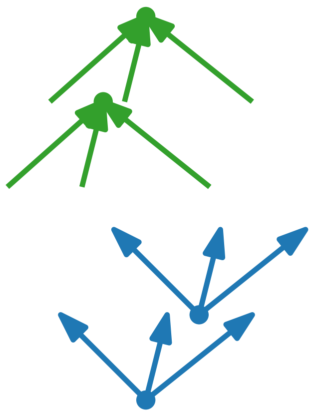
  remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$

$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$



- $G' = (V, E')$ is a DAG
  - we create an order on V
- $E \setminus E'$ is a feedback arc set
- Time: $\mathcal{O}(|V| + |E|)$

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

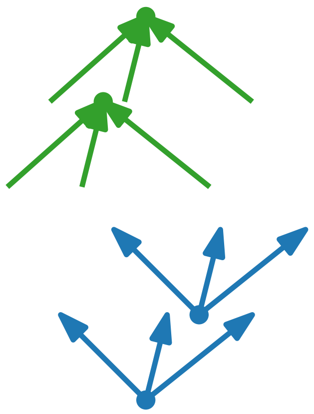
  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$



- $G' = (V, E')$ is a DAG
  - we create an order on V
- $E \setminus E'$ is a feedback arc set

- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq$

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

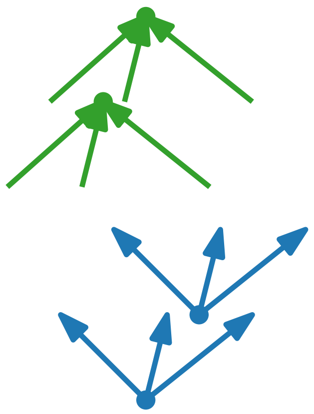
  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$



- $G' = (V, E')$ is a DAG
  - we create an order on V
- $E \setminus E'$ is a feedback arc set

- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq |E|/2$

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

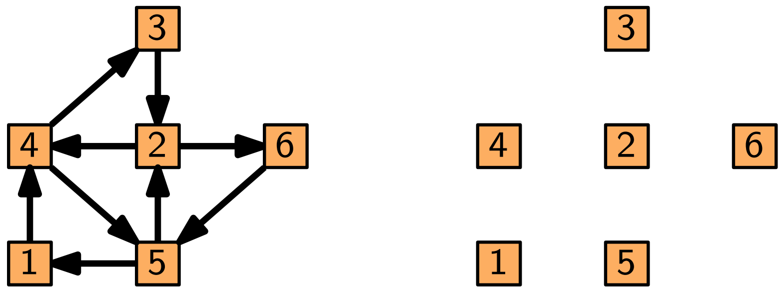
$E' \leftarrow E' \cup N^{\rightarrow}(v)$

else

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N(v)$ from G.

return (V, E')



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

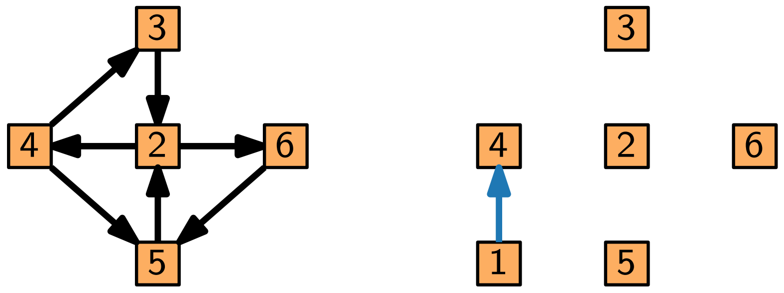
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')

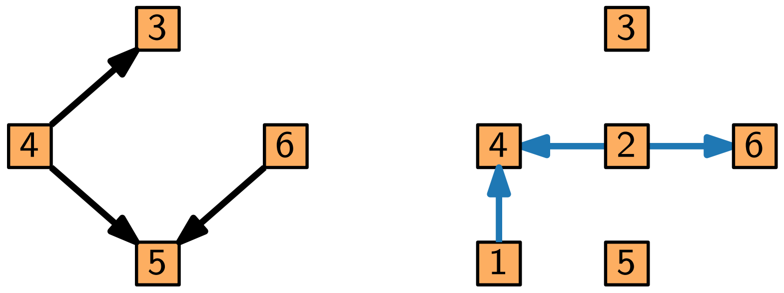


Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$
foreach $v \in V$ **do**
  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  else
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  remove v and $N(v)$ from G.
return (V, E')



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

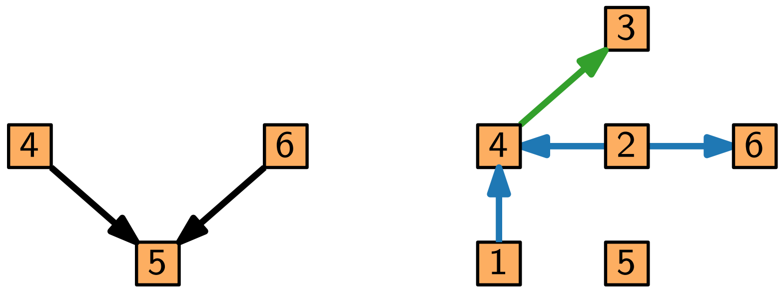
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')

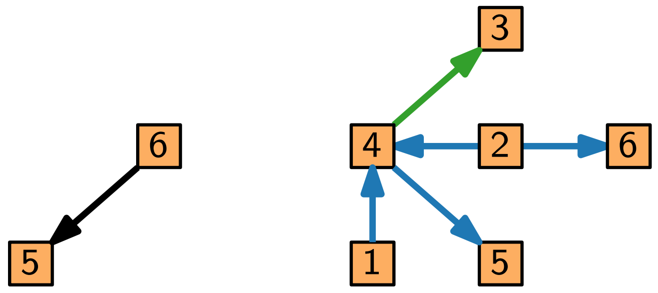


Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$
foreach $v \in V$ **do**
  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  else
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  remove v and $N(v)$ from G.
return (V, E')



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

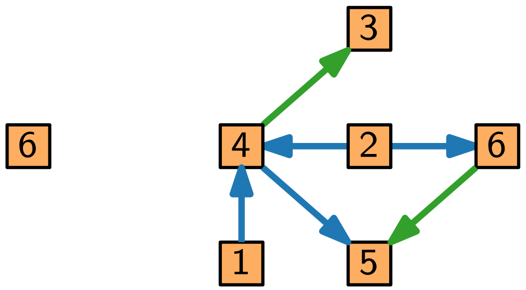
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')



Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

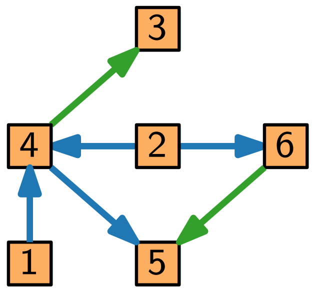
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

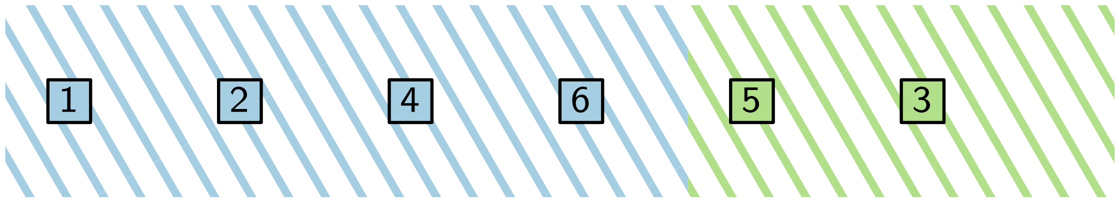
  else

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

return (V, E')





Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

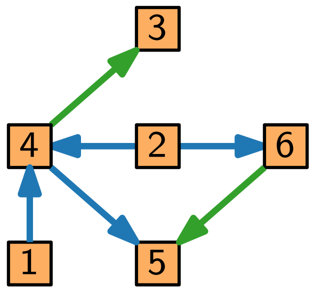
$E' \leftarrow E' \cup N^{\rightarrow}(v)$

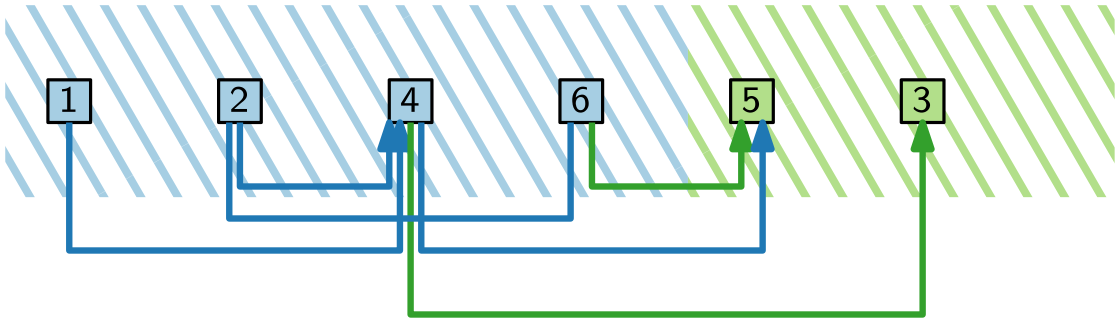
else

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N(v)$ from G.

return (V, E')





Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

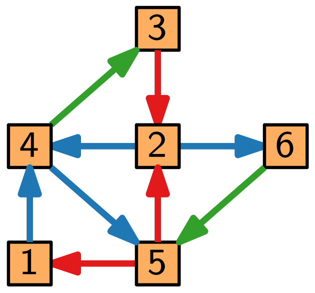
$E' \leftarrow E' \cup N^{\rightarrow}(v)$

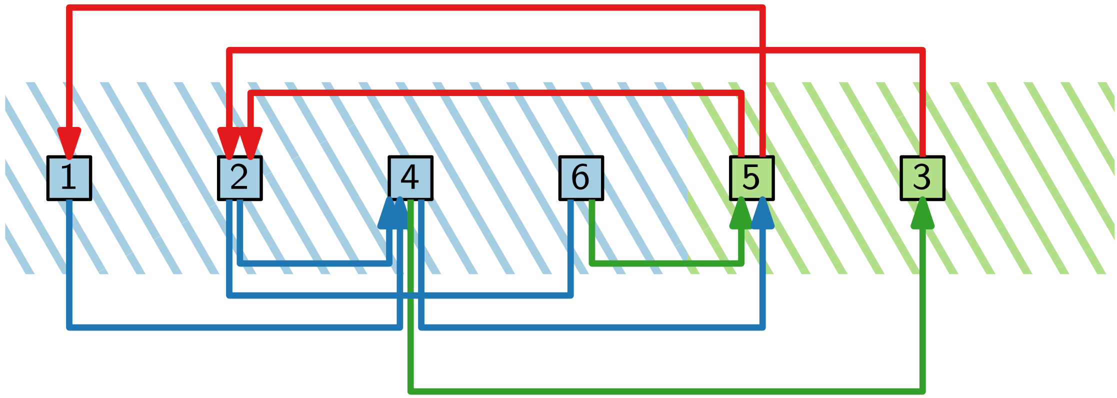
else

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N(v)$ from G.

return (V, E')





Heuristic 2

[Eades, Lin, Smyth '93]

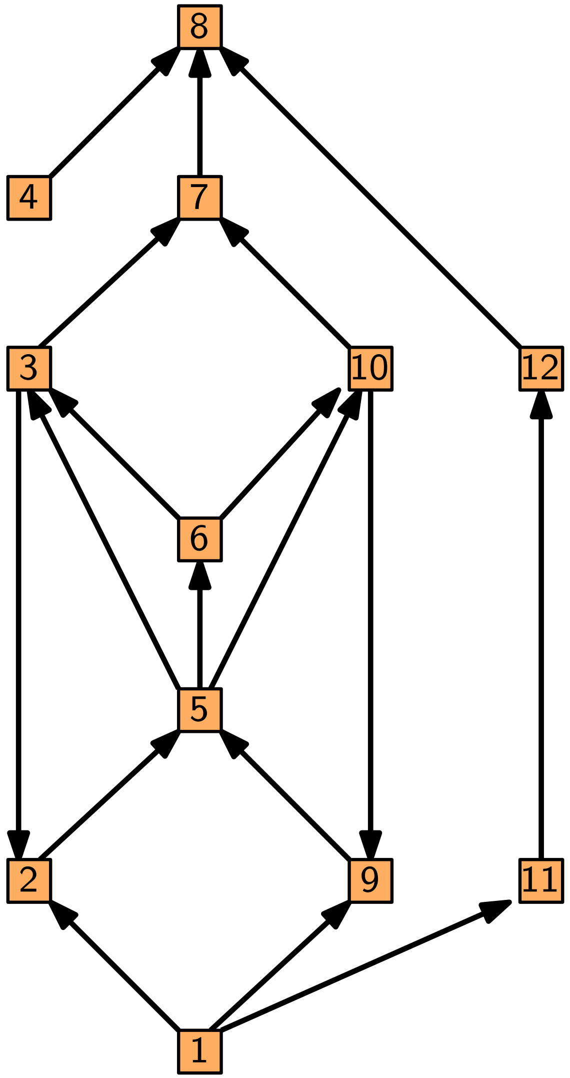
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

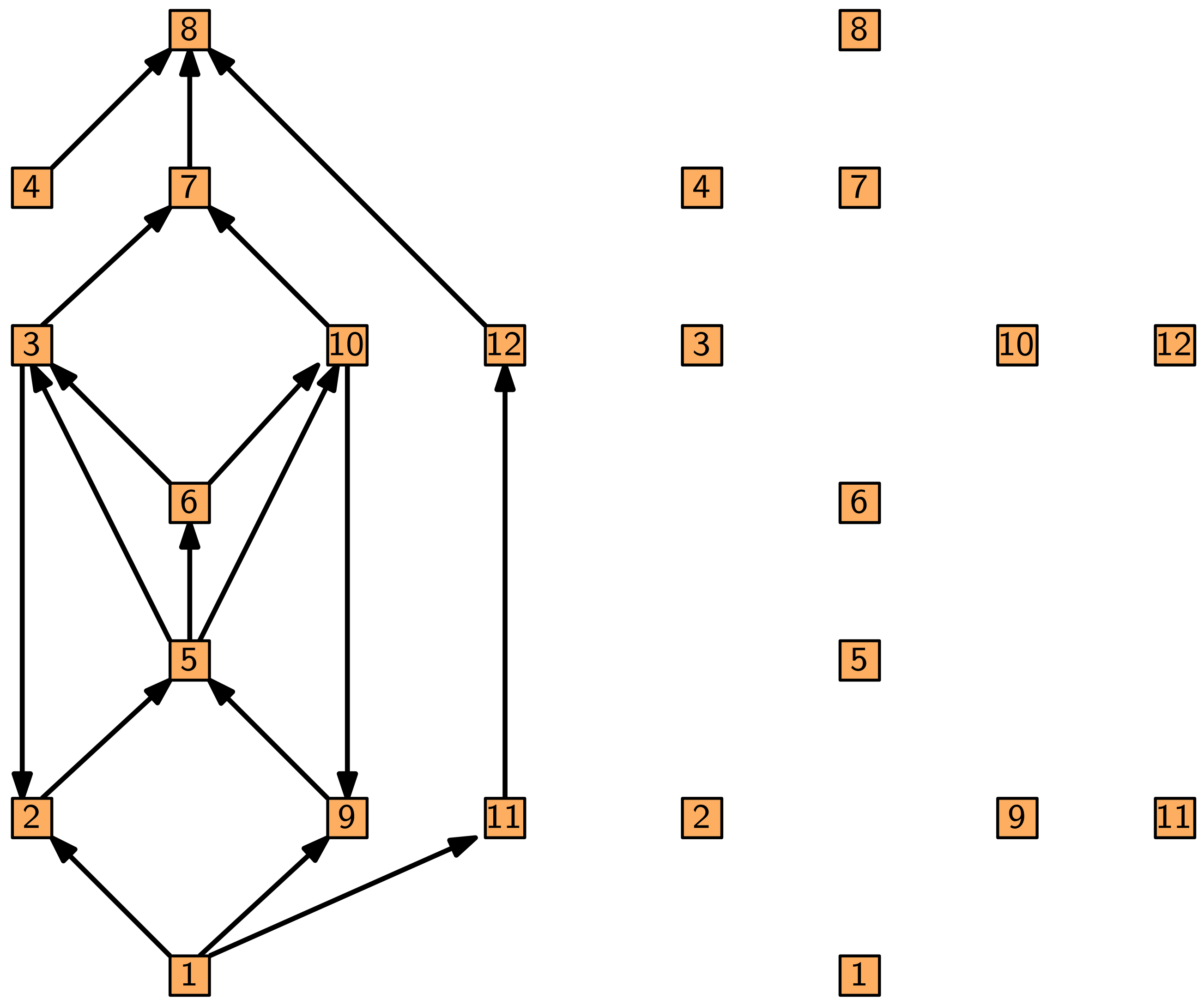
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

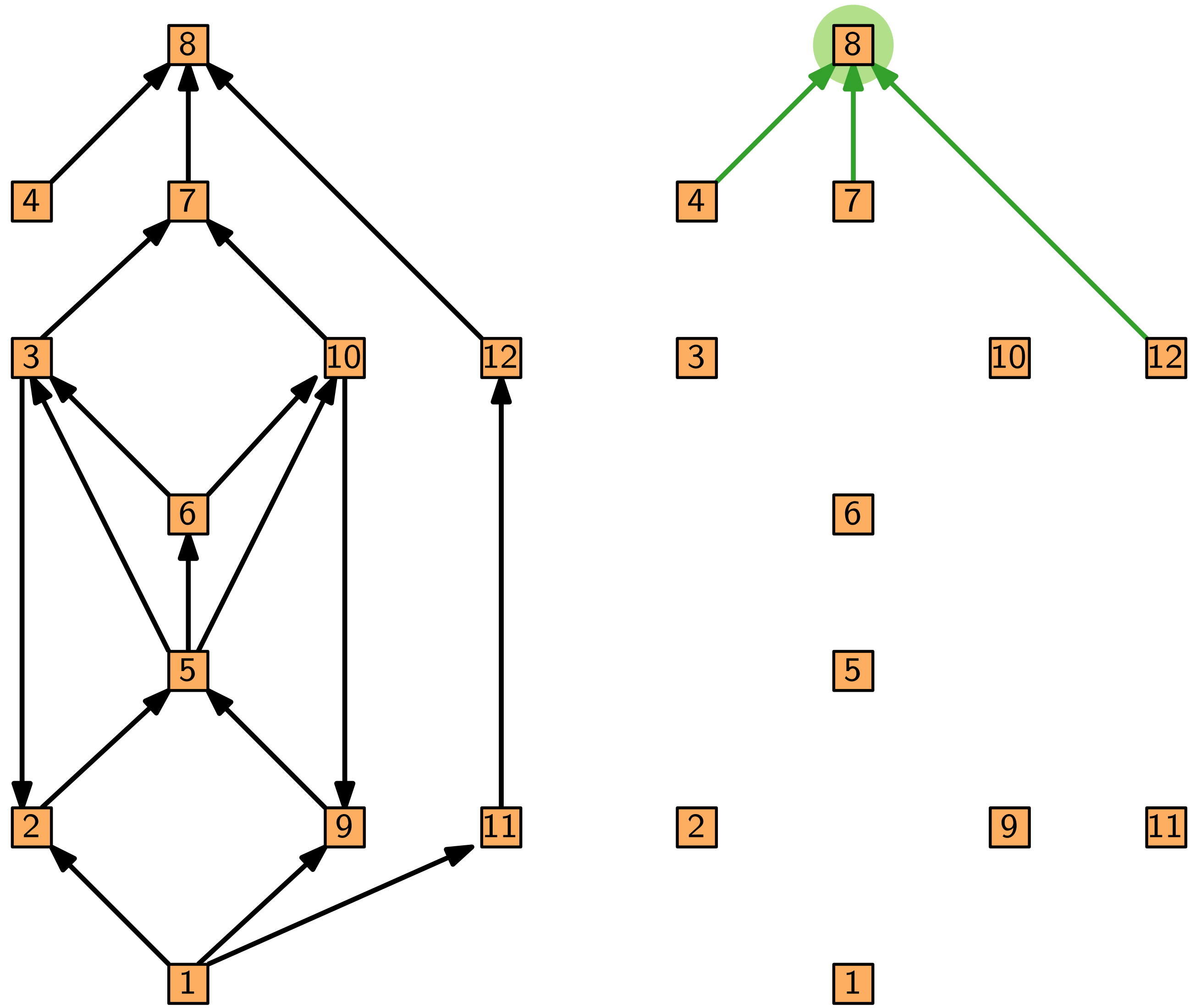
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

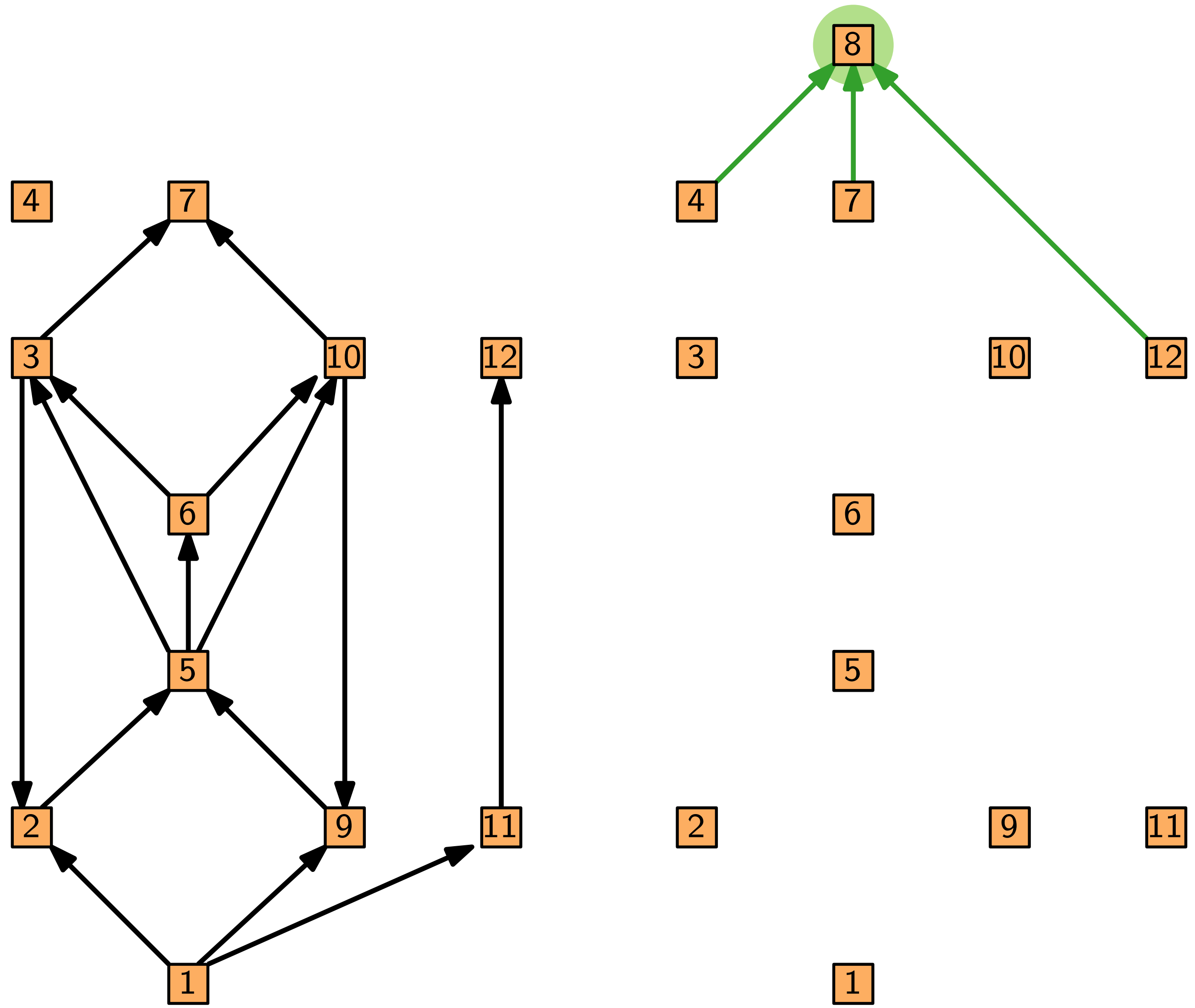
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

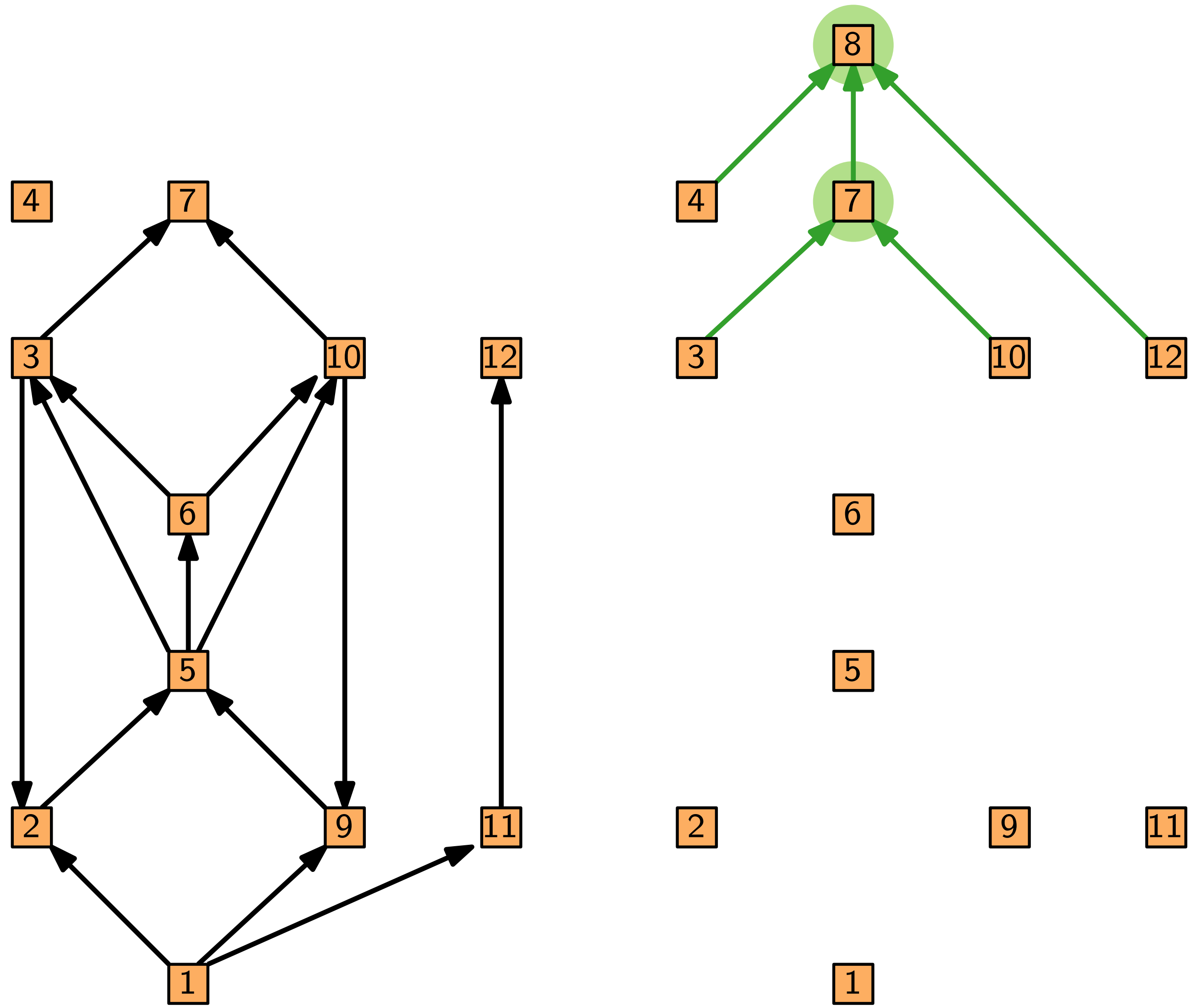
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

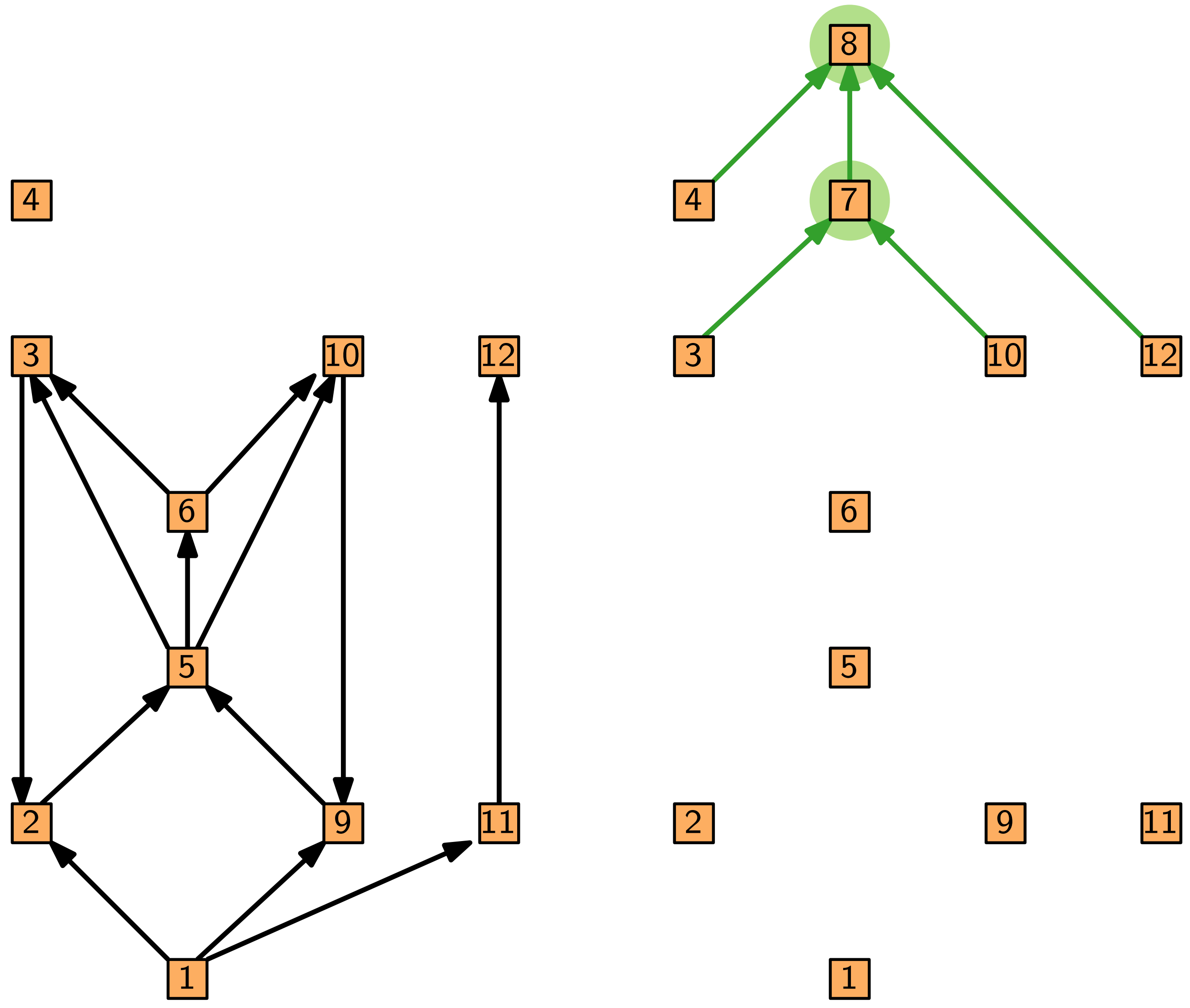
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

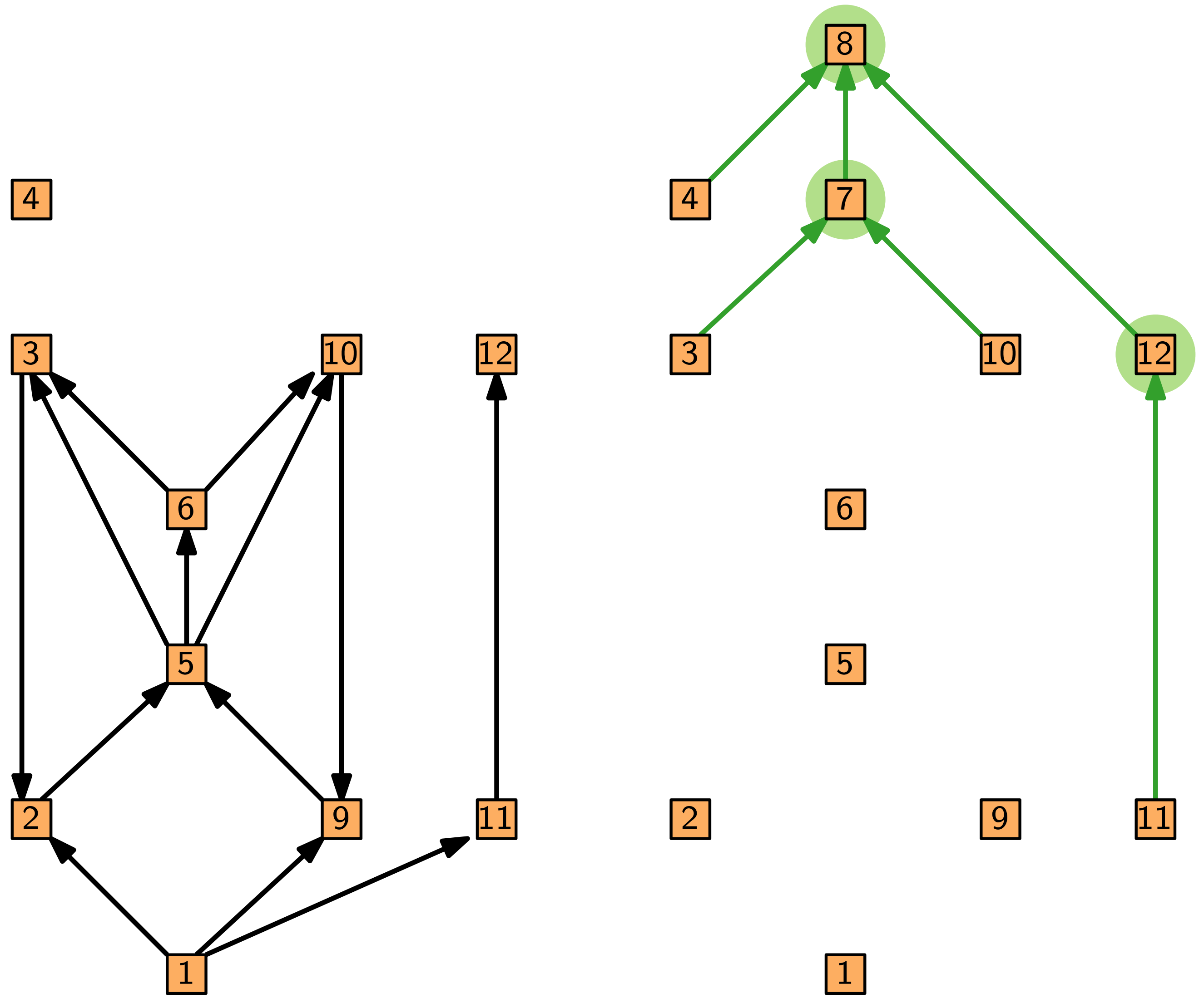
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

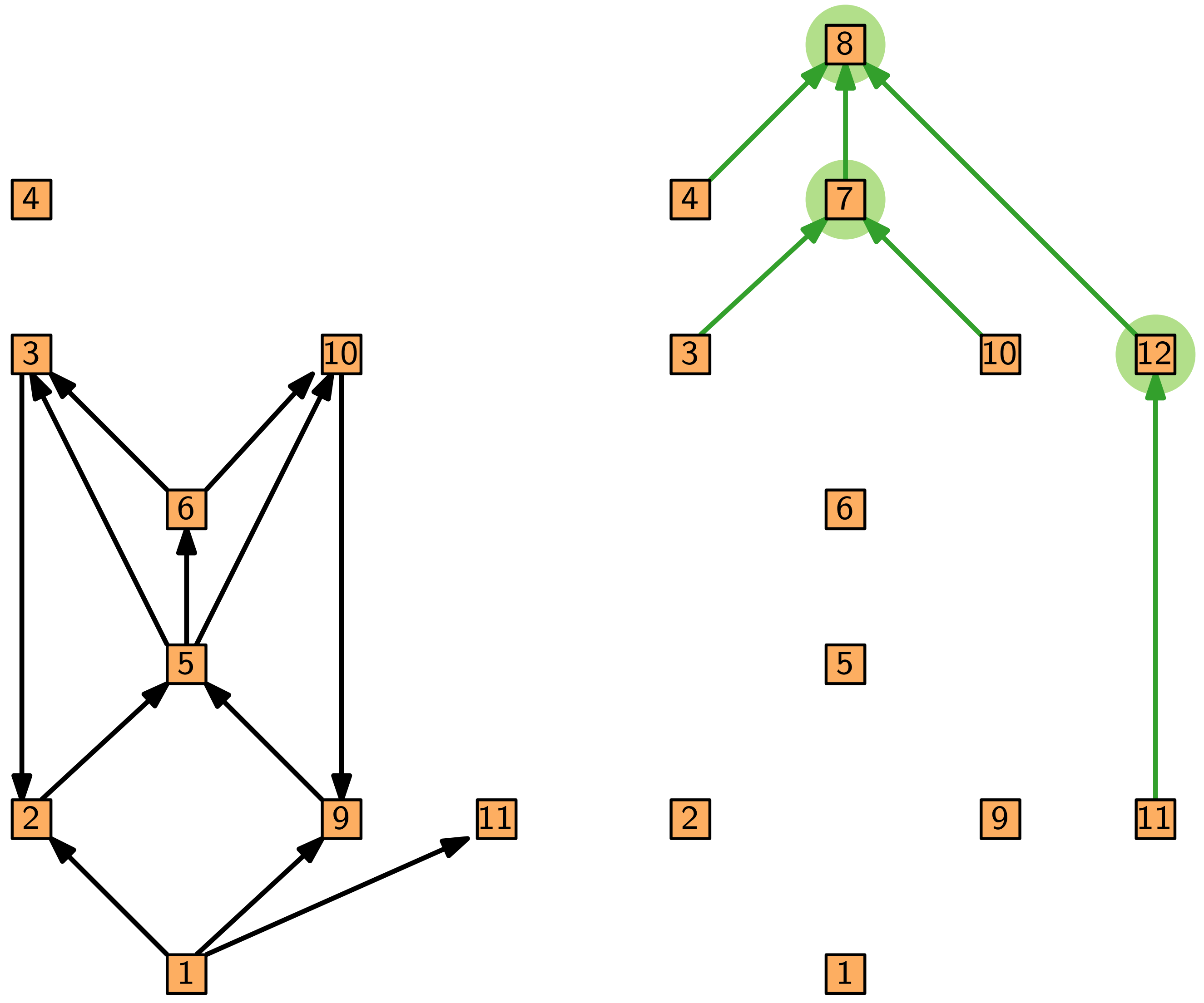
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

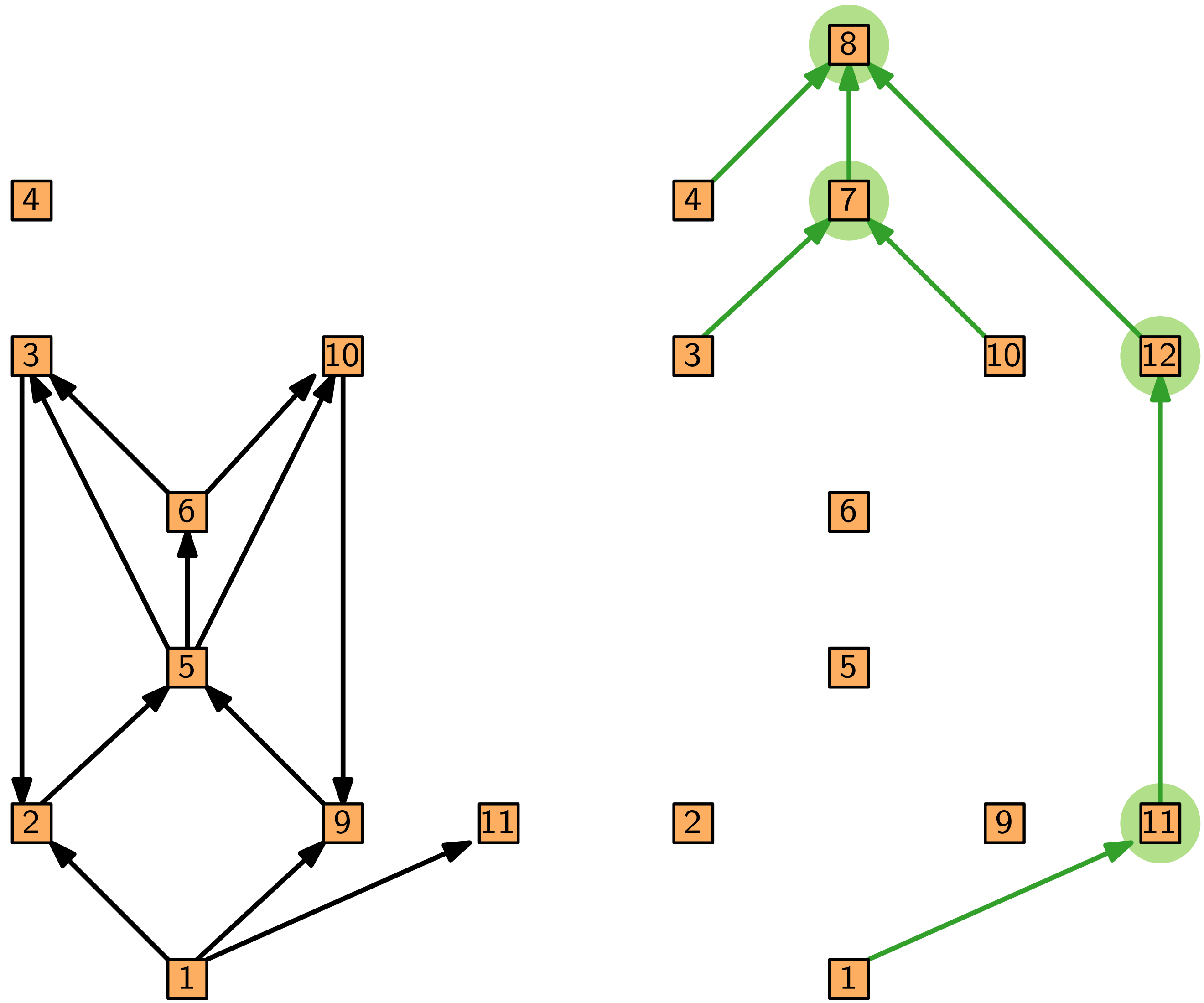
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

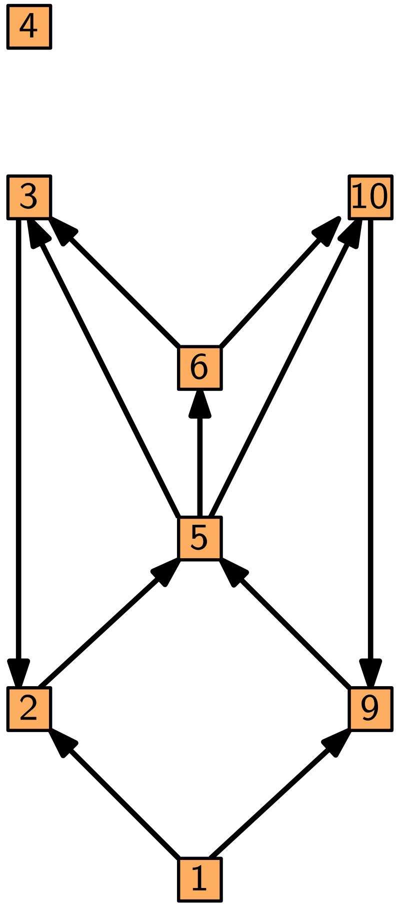
$E' \leftarrow \emptyset$

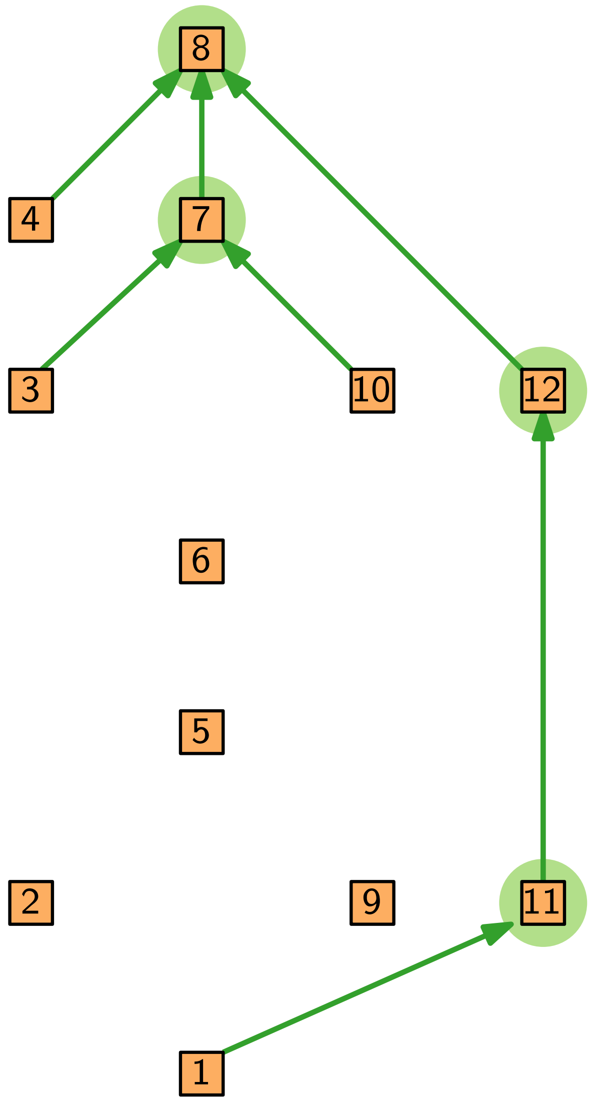
while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$





Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

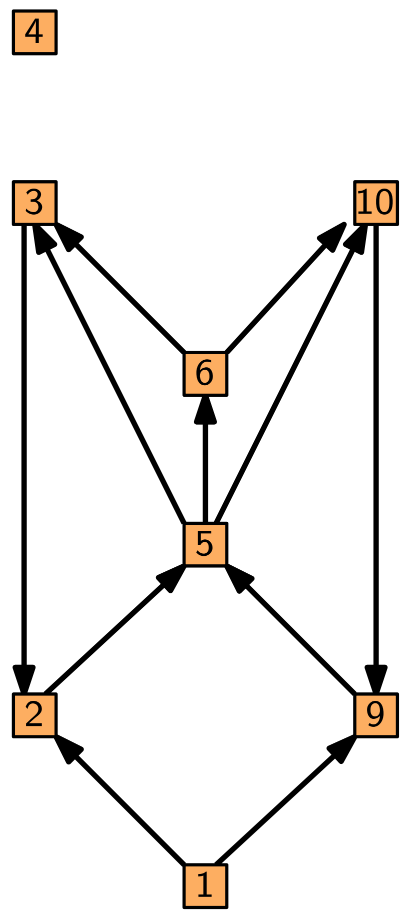
while $V \neq \emptyset$ **do**

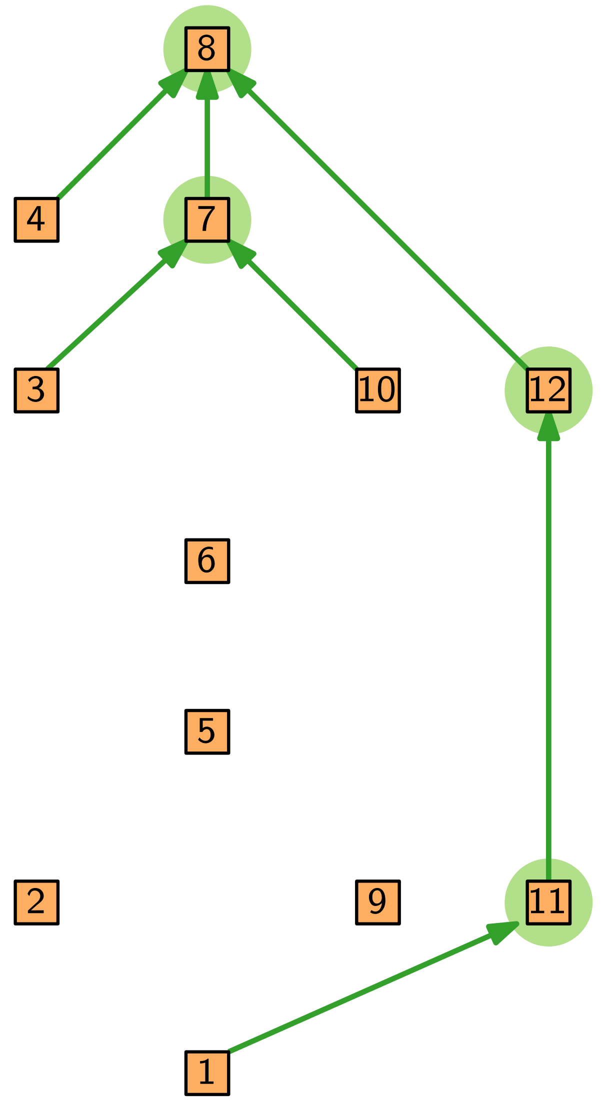
  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$

  Remove all isolated vertices from V





Heuristic 2

[Eades, Lin, Smyth ’93]

$E' \leftarrow \emptyset$

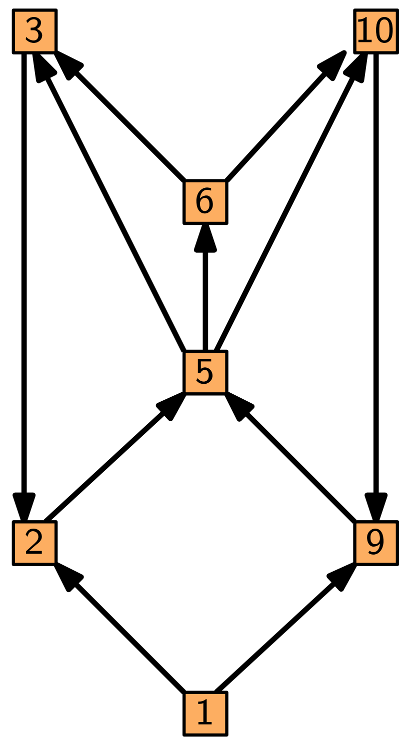
while $V \neq \emptyset$ **do**

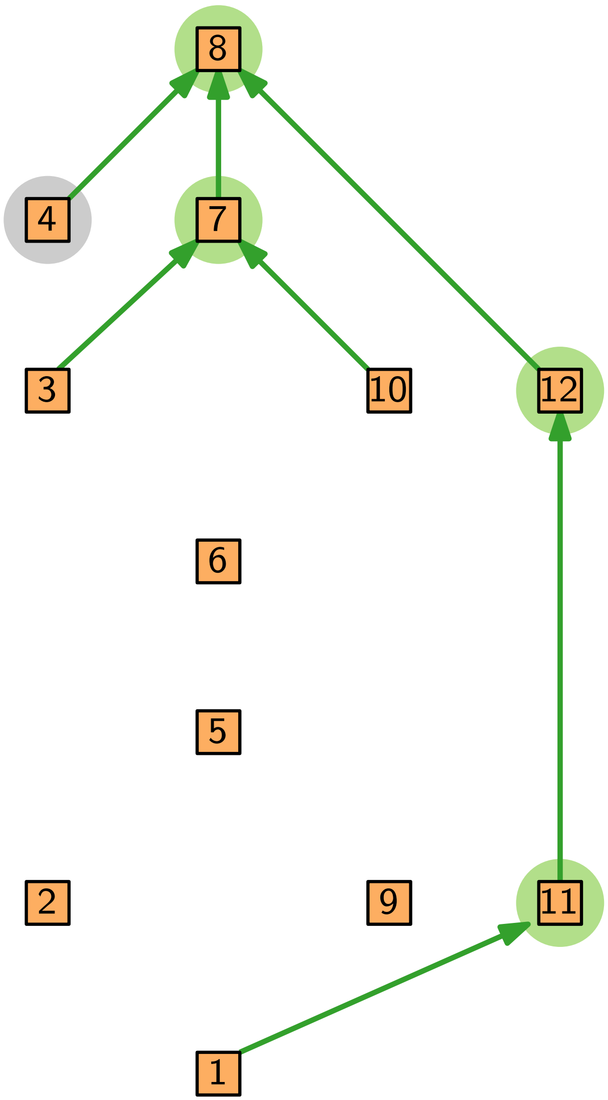
  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$

  Remove all isolated vertices from V





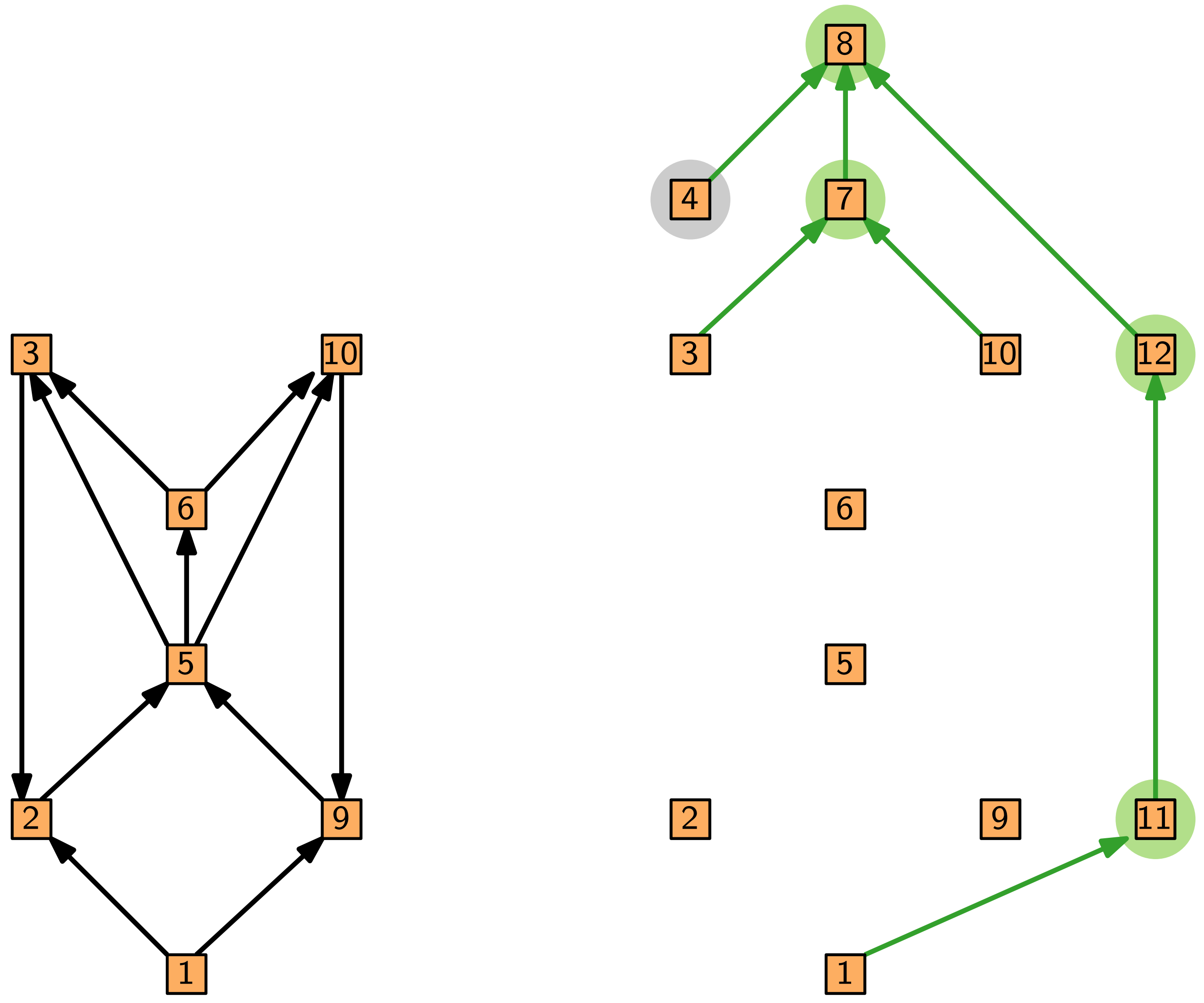
Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

- **while** in V exists a sink v **do**
  - $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  - remove v and $N^{\leftarrow}(v)$
- Remove all isolated vertices from V
- **while** in V exists a source v **do**
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N^{\rightarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

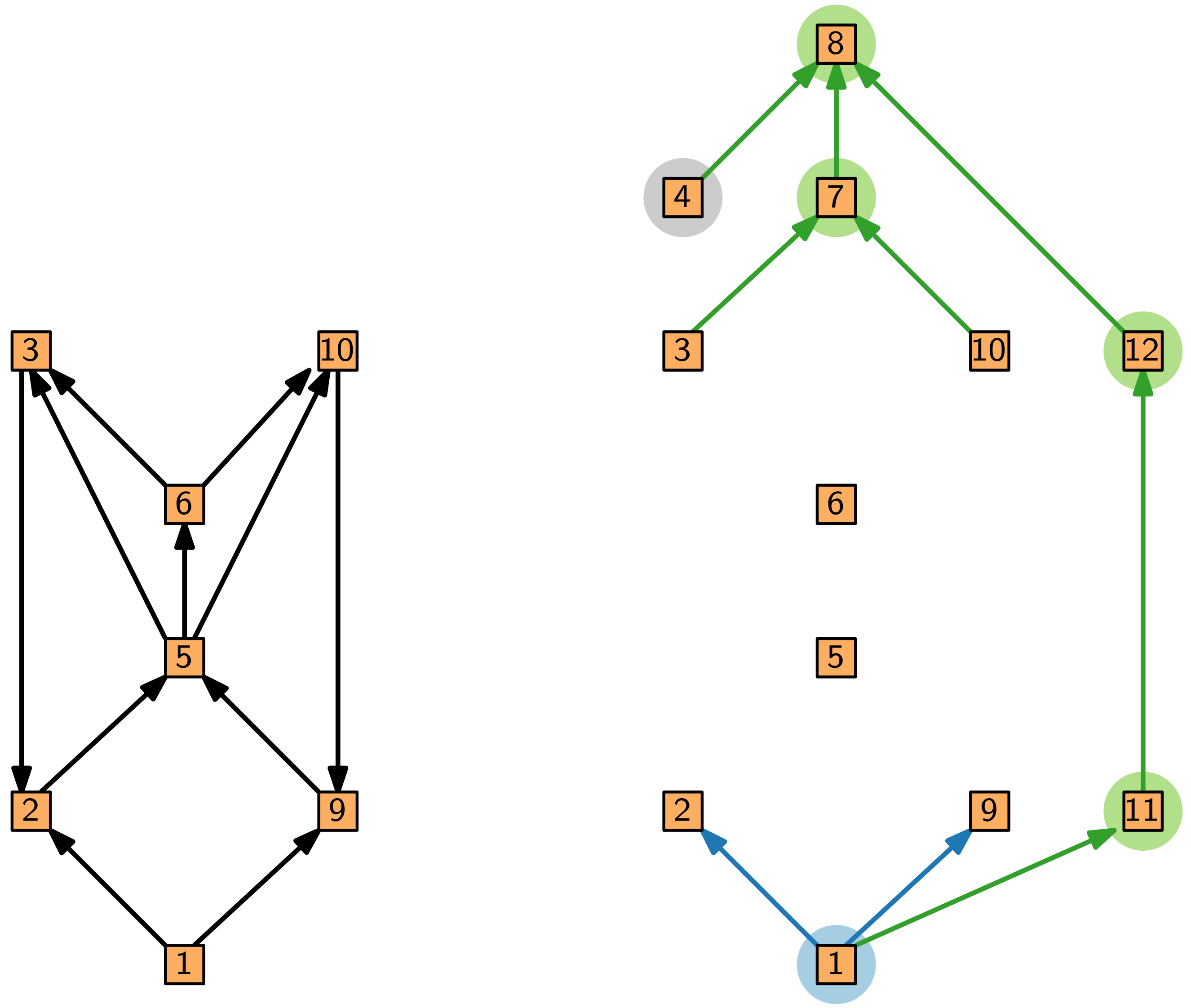
    remove v and $N^{\leftarrow}(v)$

  Remove all isolated vertices from V

  while in V exists a source v **do**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

    remove v and $N^{\rightarrow}(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

while in V exists a sink v **do**

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

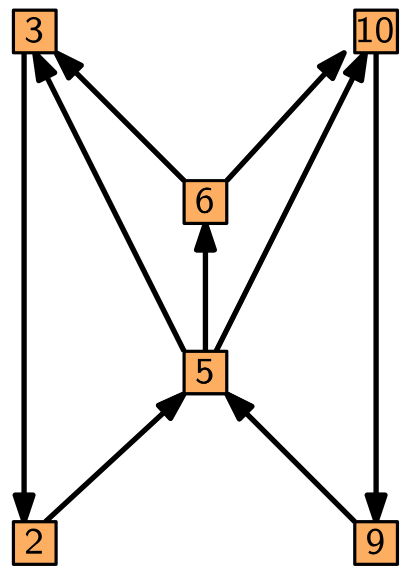
remove v and $N^{\leftarrow}(v)$

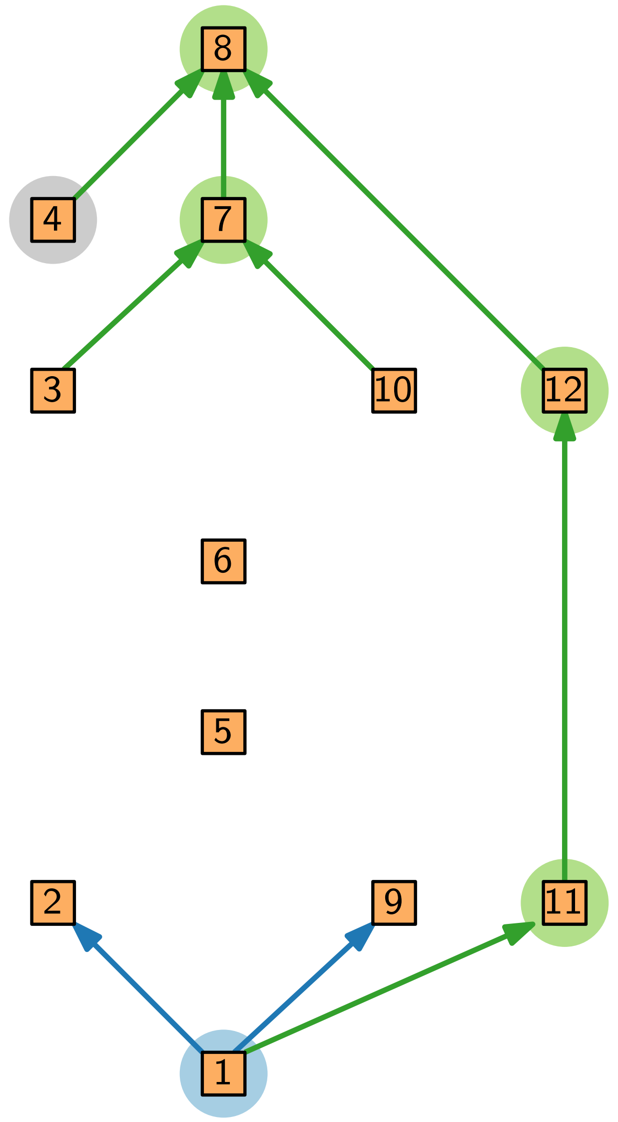
Remove all isolated vertices from V

while in V exists a source v **do**

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

remove v and $N^{\rightarrow}(v)$

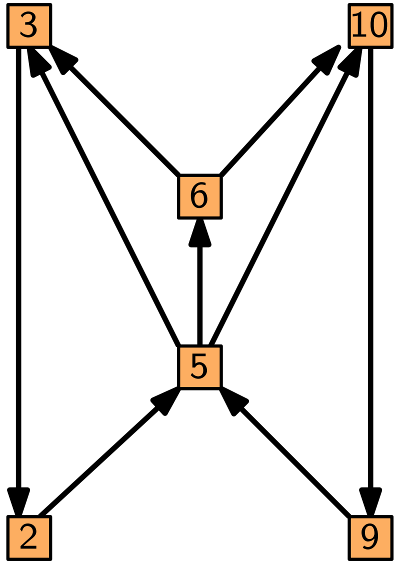


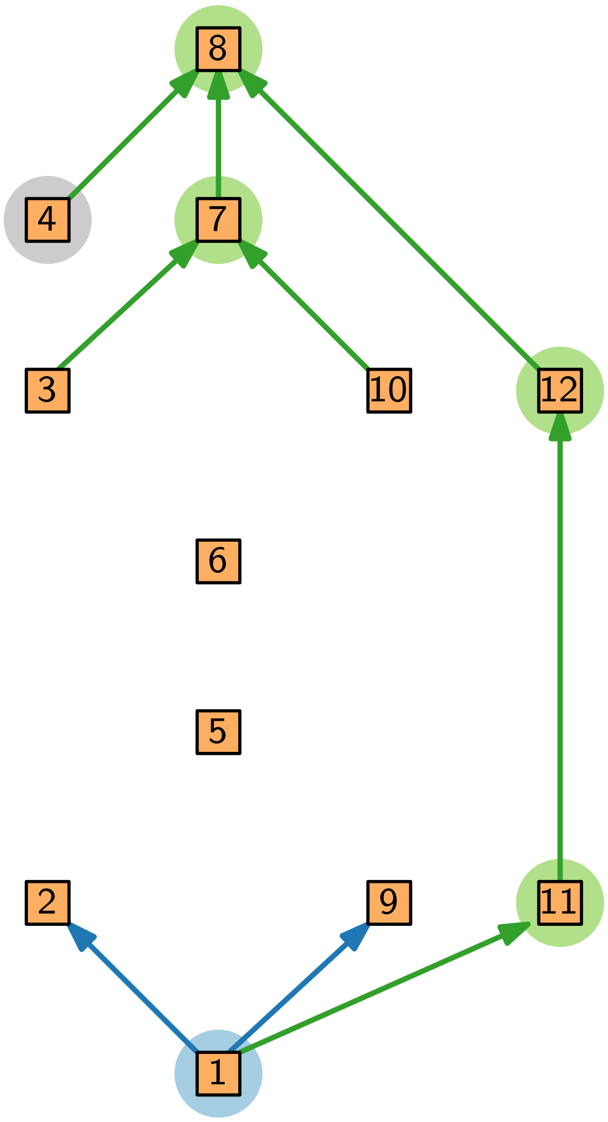


Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$
while $V \neq \emptyset$ **do**
  while in V exists a sink v **do**
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
    remove v and $N^{\leftarrow}(v)$
  Remove all isolated vertices from V
  while in V exists a source v **do**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N^{\rightarrow}(v)$
  if $V \neq \emptyset$ **then**
    let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N(v)$

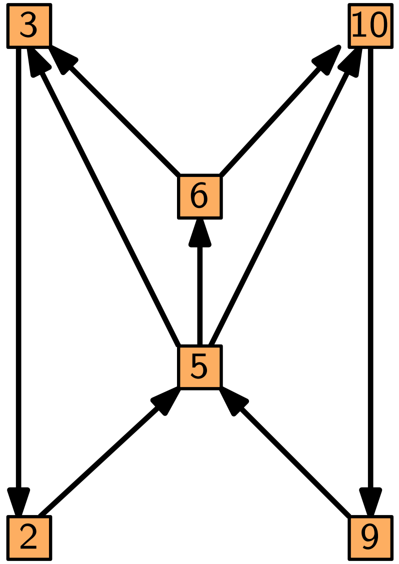


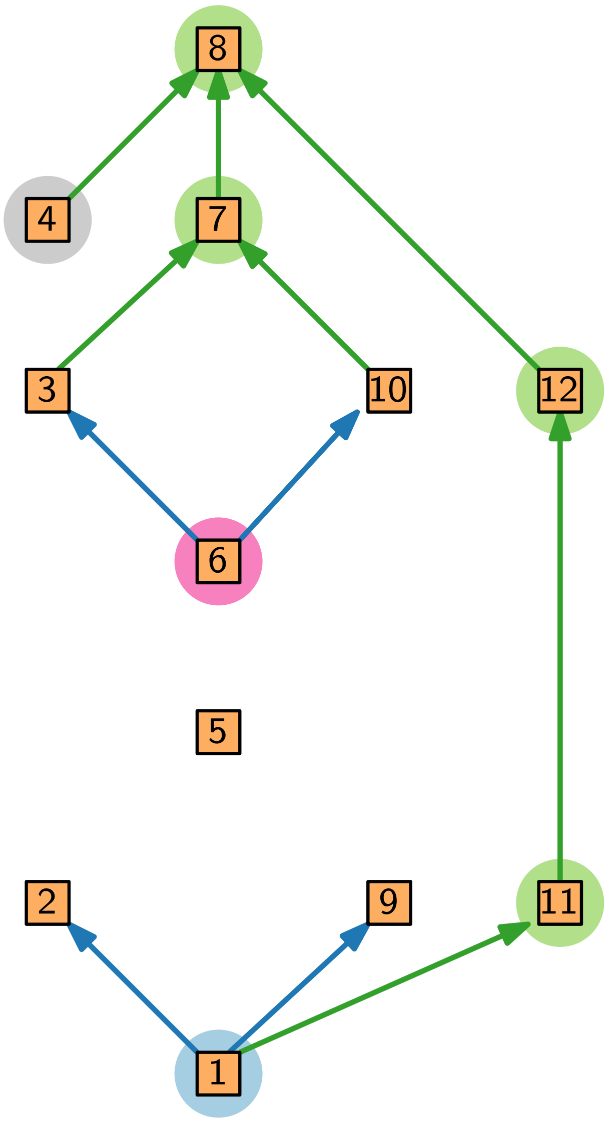


Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$
while $V \neq \emptyset$ **do**
  while in V exists a sink v **do**
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
    remove v and $N^{\leftarrow}(v)$
  Remove all isolated vertices from V
  while in V exists a source v **do**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N^{\rightarrow}(v)$
  if $V \neq \emptyset$ **then**
    let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N(v)$





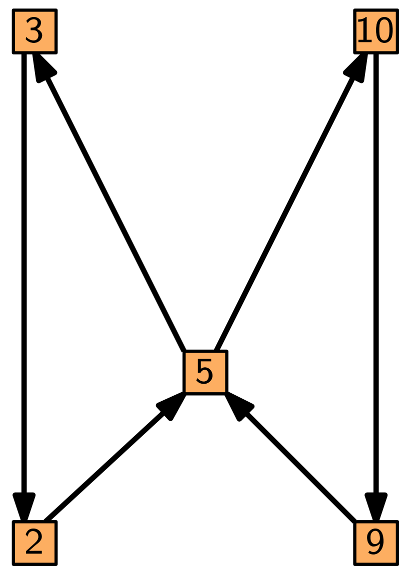
Heuristic 2

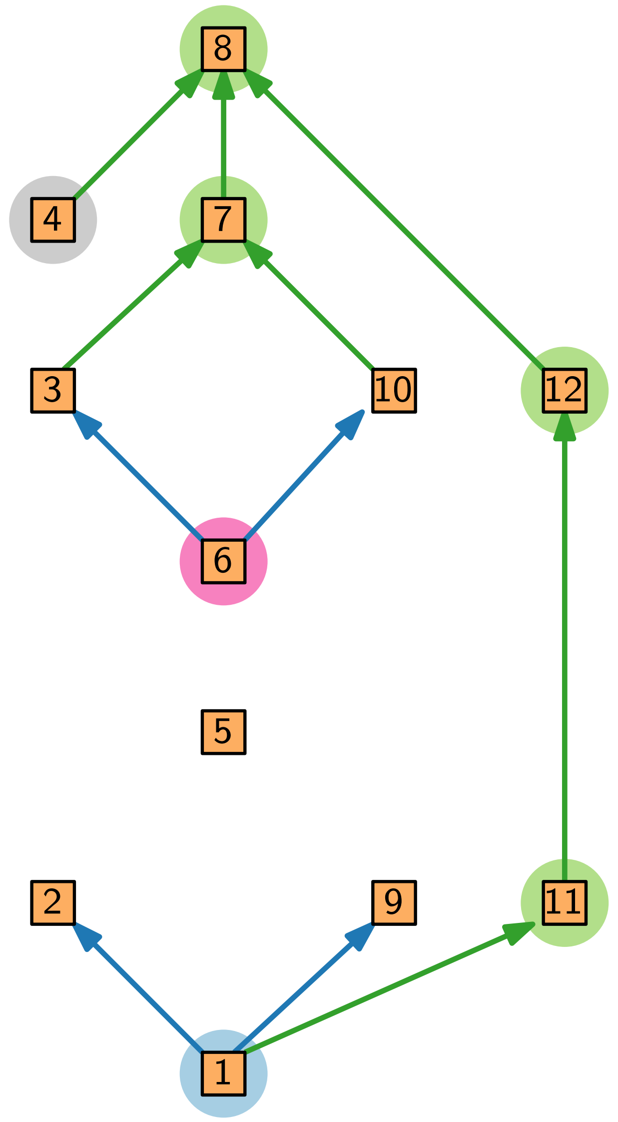
[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

- **while** in V exists a sink v **do**
  - $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  - remove v and $N^{\leftarrow}(v)$
- Remove all isolated vertices from V
- **while** in V exists a source v **do**
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N^{\rightarrow}(v)$
- **if** $V \neq \emptyset$ **then**
  - let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N(v)$





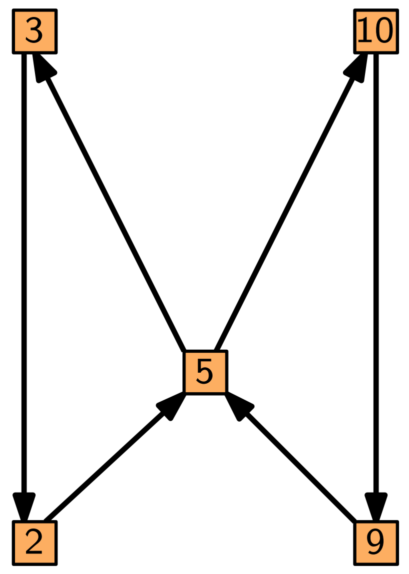
Heuristic 2

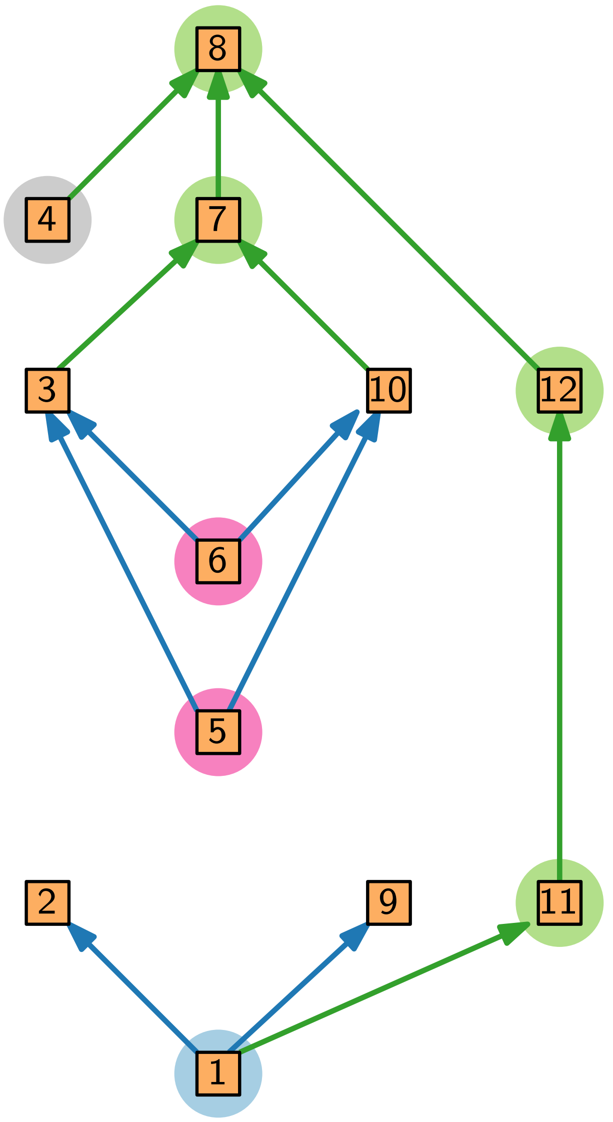
[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

- **while** in V exists a sink v **do**
  - $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  - remove v and $N^{\leftarrow}(v)$
- Remove all isolated vertices from V
- **while** in V exists a source v **do**
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N^{\rightarrow}(v)$
- **if** $V \neq \emptyset$ **then**
  - let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N(v)$

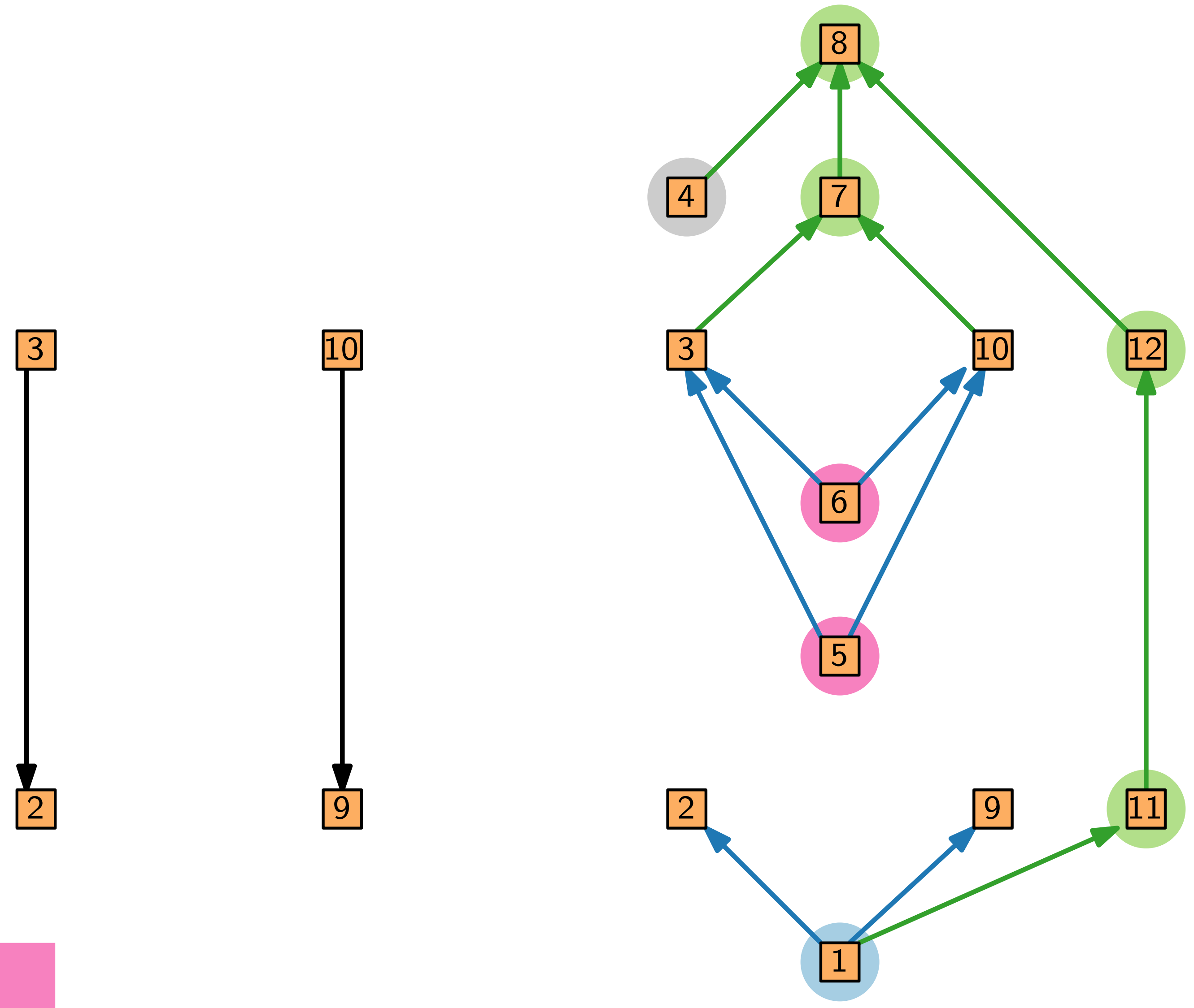




Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$
while $V \neq \emptyset$ **do**
  while in V exists a sink v **do**
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
    remove v and $N^{\leftarrow}(v)$
  Remove all isolated vertices from V
  while in V exists a source v **do**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N^{\rightarrow}(v)$
  if $V \neq \emptyset$ **then**
    let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

while in V exists a sink v **do**

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N^{\leftarrow}(v)$

Remove all isolated vertices from V

while in V exists a source v **do**

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

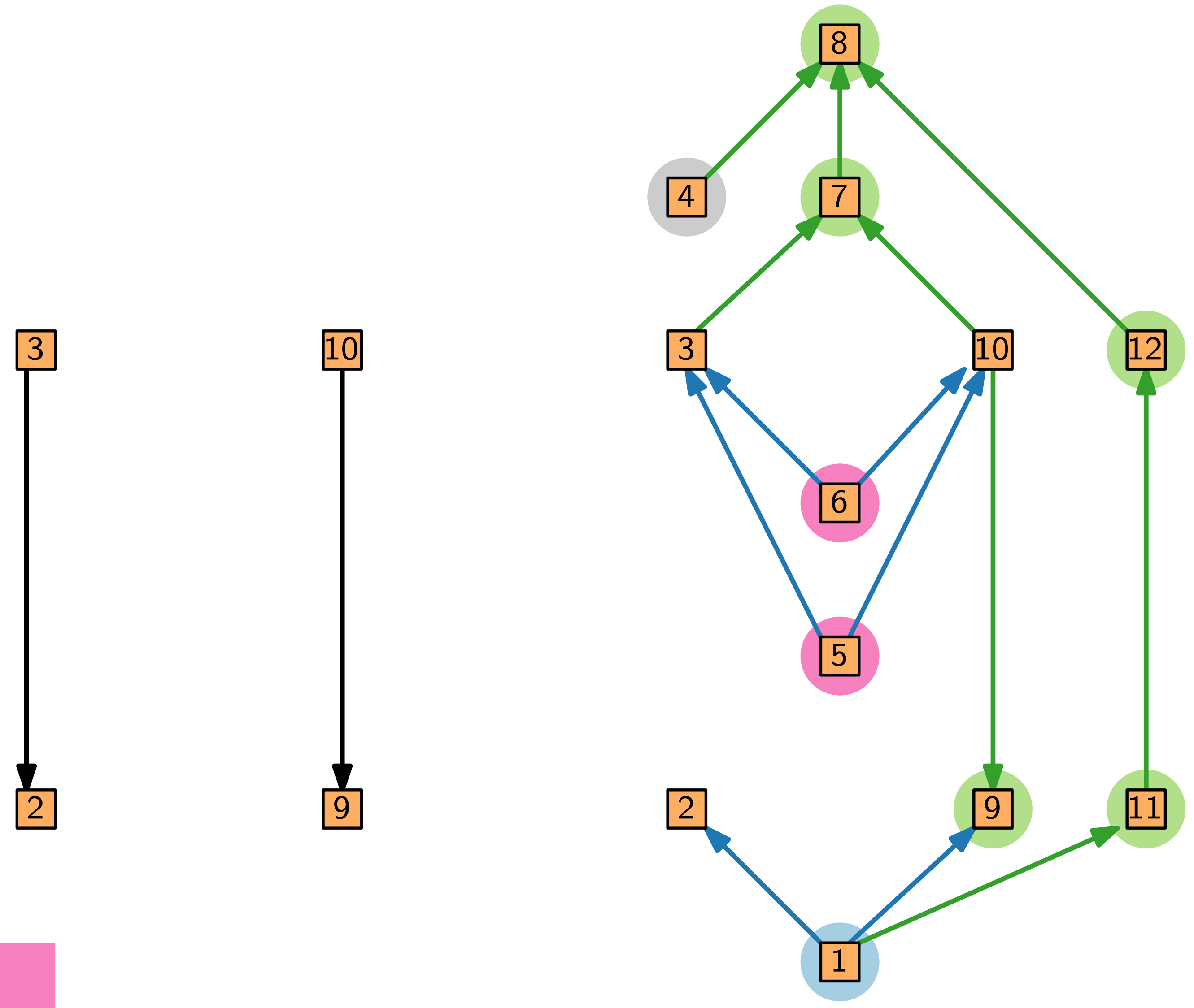
remove v and $N^{\rightarrow}(v)$

if $V \neq \emptyset$ **then**

let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

remove v and $N(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

while in V exists a sink v **do**

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N^{\leftarrow}(v)$

Remove all isolated vertices from V

while in V exists a source v **do**

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

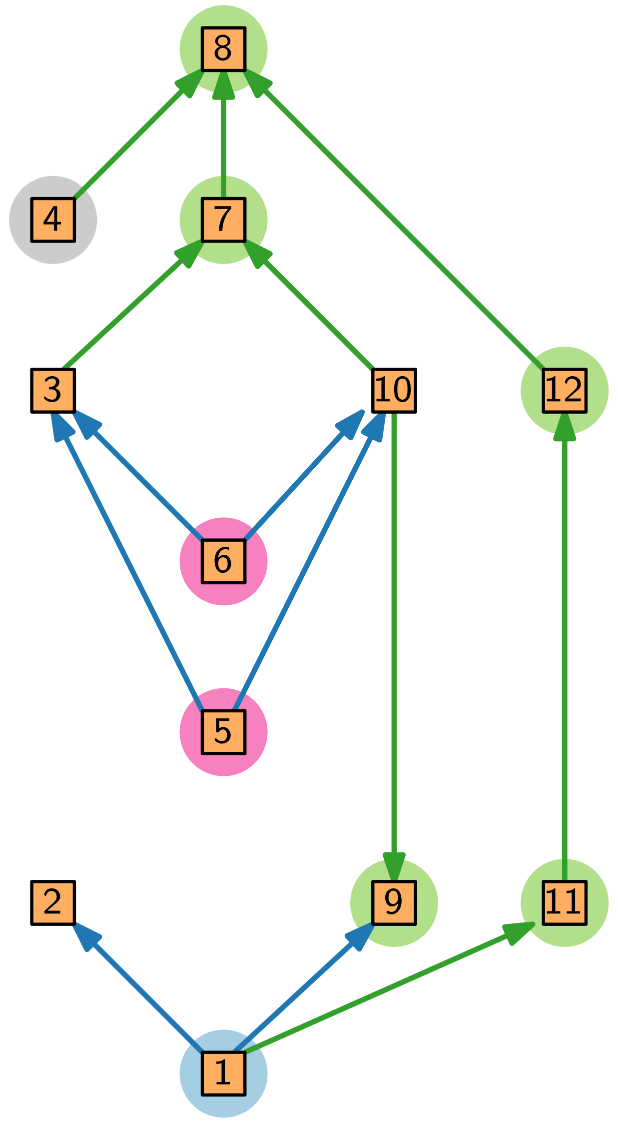
remove v and $N^{\rightarrow}(v)$

if $V \neq \emptyset$ **then**

let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

remove v and $N(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$

  Remove all isolated vertices from V

  while in V exists a source v **do**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

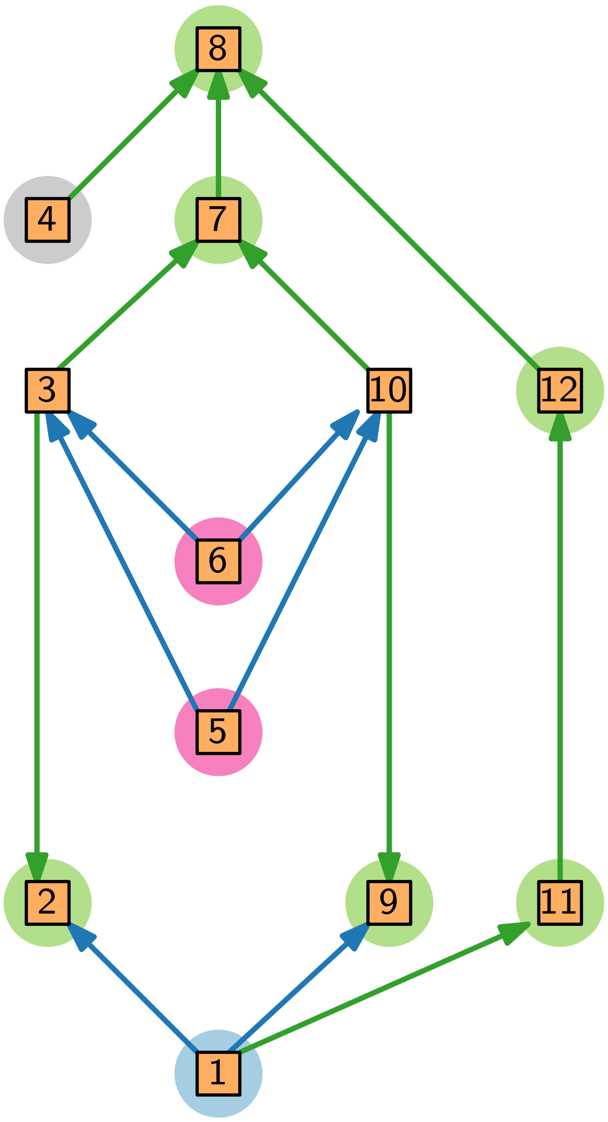
    remove v and $N^{\rightarrow}(v)$

  if $V \neq \emptyset$ **then**

    let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

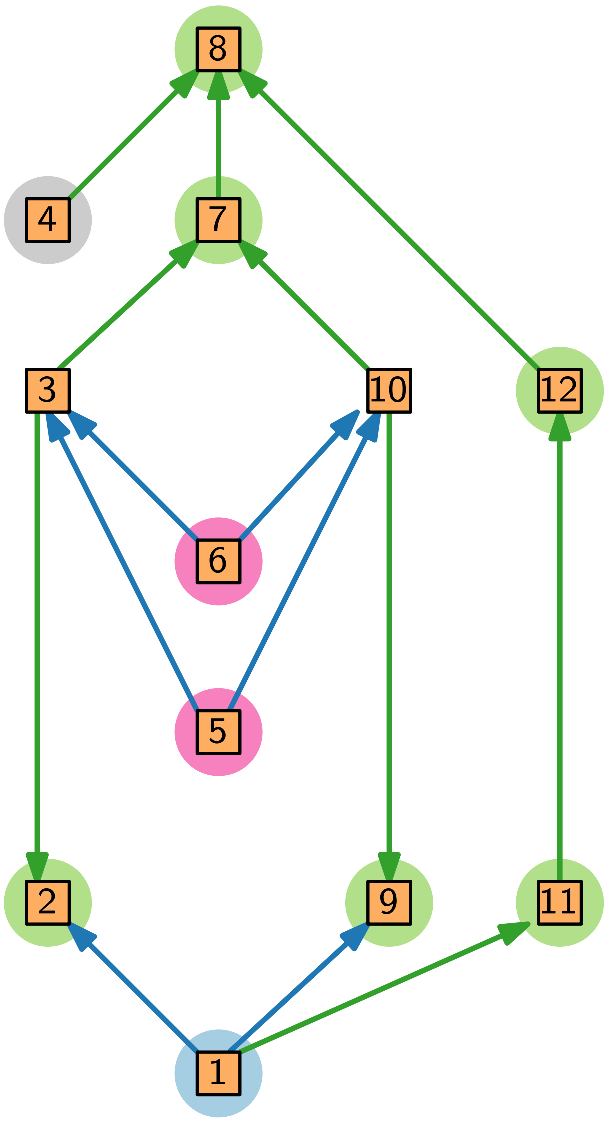
    remove v and $N(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$
while $V \neq \emptyset$ **do**
- **while** in V exists a sink v **do**
  - $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  - remove v and $N^{\leftarrow}(v)$
- Remove all isolated vertices from V
- **while** in V exists a source v **do**
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N^{\rightarrow}(v)$
- **if** $V \neq \emptyset$ **then**
  - let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

while in V exists a sink v **do**

$E' \leftarrow E' \cup N^{\leftarrow}(v)$

remove v and $N^{\leftarrow}(v)$

Remove all isolated vertices from V

while in V exists a source v **do**

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

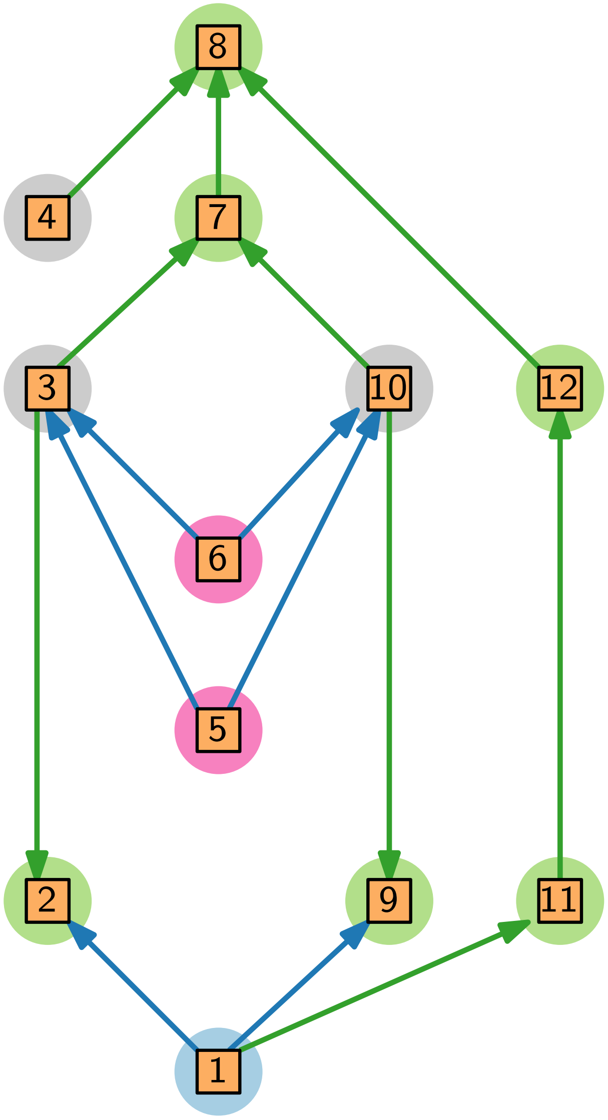
remove v and $N^{\rightarrow}(v)$

if $V \neq \emptyset$ **then**

let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;

$E' \leftarrow E' \cup N^{\rightarrow}(v)$

remove v and $N(v)$



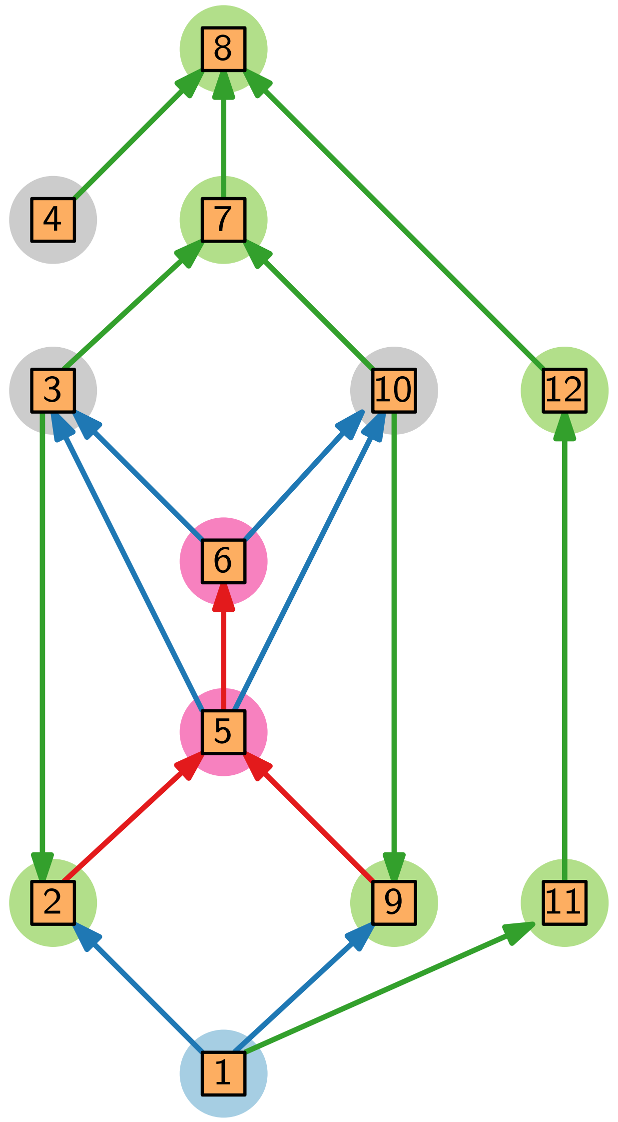
Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

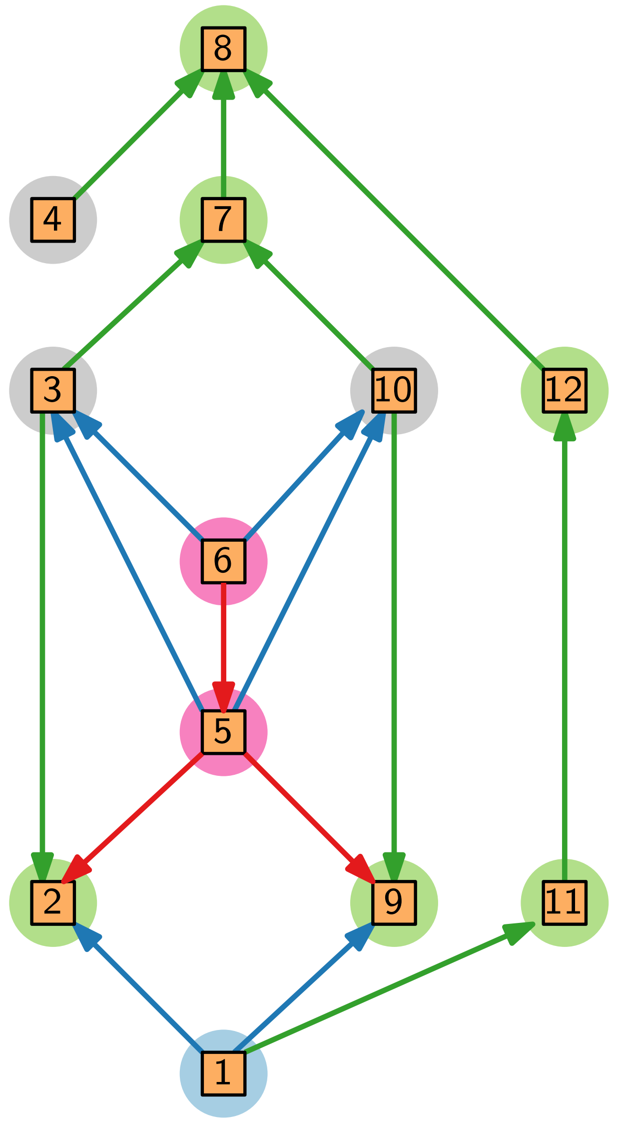
- **while** in V exists a sink v **do**
  - $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  - remove v and $N^{\leftarrow}(v)$
- Remove all isolated vertices from V
- **while** in V exists a source v **do**
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N^{\rightarrow}(v)$
- **if** $V \neq \emptyset$ **then**
  - let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N(v)$



Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$
while $V \neq \emptyset$ **do**
  while in V exists a sink v **do**
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
    remove v and $N^{\leftarrow}(v)$
  Remove all isolated vertices from V
  while in V exists a source v **do**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N^{\rightarrow}(v)$
  if $V \neq \emptyset$ **then**
    let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N(v)$



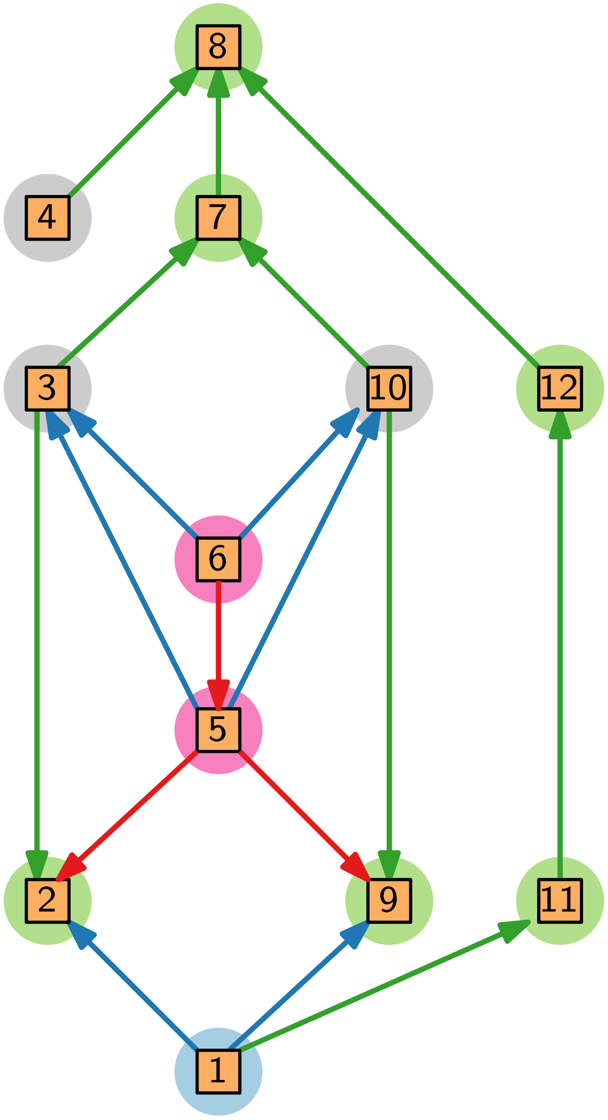
Heuristic 2

[Eades, Lin, Smyth '93]

$E' \leftarrow \emptyset$

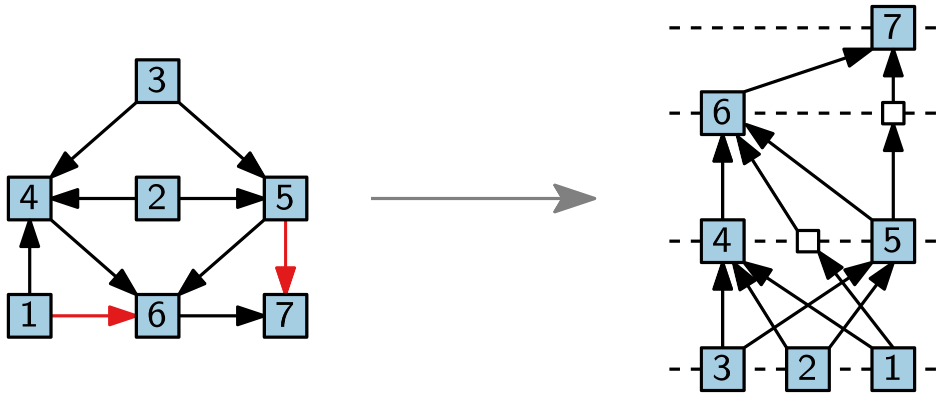
while $V \neq \emptyset$ **do**

- **while** in V exists a sink v **do**
  - $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  - remove v and $N^{\leftarrow}(v)$
- Remove all isolated vertices from V
- **while** in V exists a source v **do**
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N^{\rightarrow}(v)$
- **if** $V \neq \emptyset$ **then**
  - let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal;
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N(v)$



- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq |E|/2 + |V|/6$

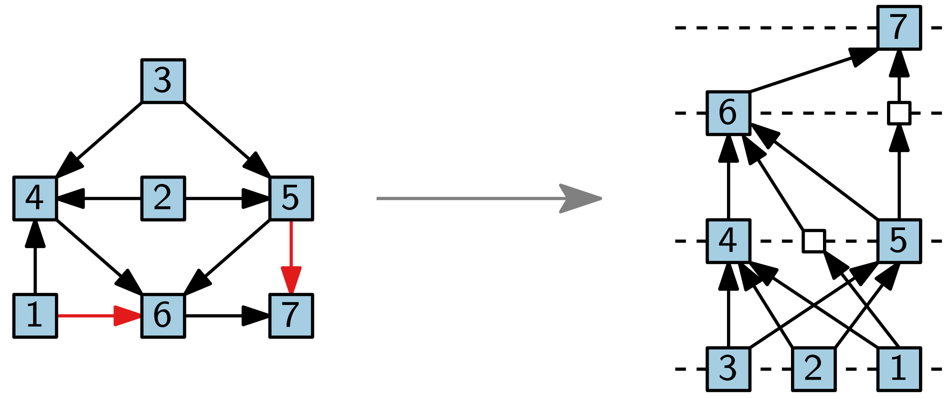
Step 2: Leveling



Problem.

- Input: acyclic, digraph $G = (V, E)$
- Output: Mapping $y \colon V \to \{1, \ldots, |V|\}$,
  so that for every $uv \in A$, $y(u) < y(v)$.

Step 2: Leveling

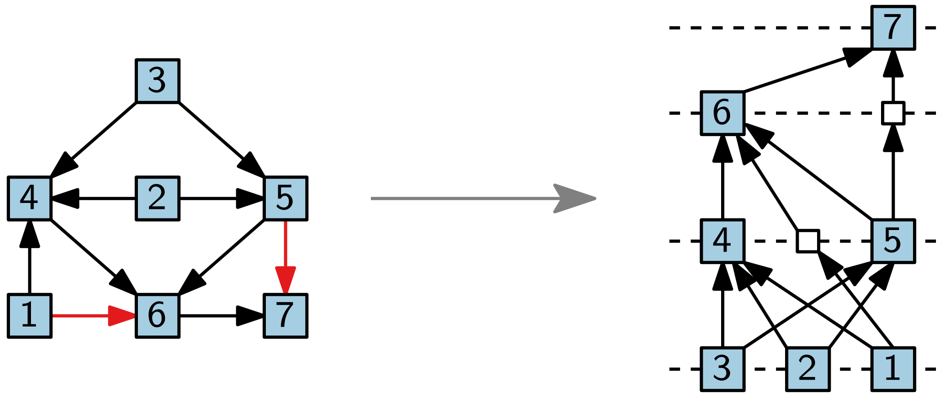


Problem.

- Input: acyclic, digraph $G = (V, E)$
- Output: Mapping $y \colon V \to \{1, \dots, |V|\}$,
  so that for every $uv \in A$, $y(u) < y(v)$.

Objective is to *minimize . . .*

Step 2: Leveling



Problem.

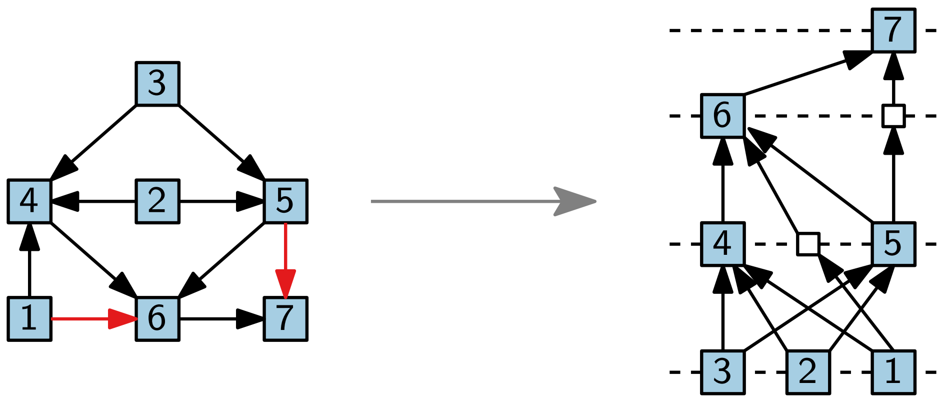
- Input: acyclic, digraph $G = (V, E)$
- Output: Mapping $y \colon V \to \{1, \ldots, |V|\}$, so that for every $uv \in A$, $y(u) < y(v)$.

Objective is to *minimize* . . .

- number of layers, i.e. $|y(V)|$
- length of the longest edge, i.e. $\max_{uv \in A} y(v) - y(u)$
- width, i.e. $\max\{|L_i| \mid 1 \leq i \leq h\}$
- total edge length, i.e. number of dummy vertices

Min number of layers

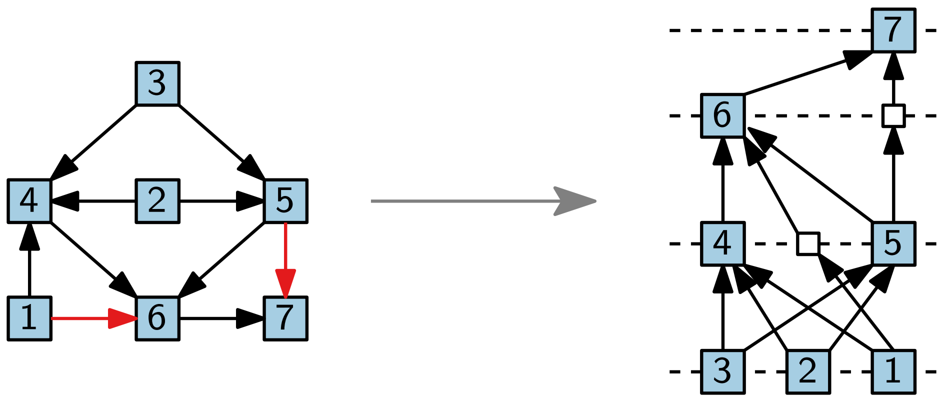
Algorithm.



Min number of layers

Algorithm.

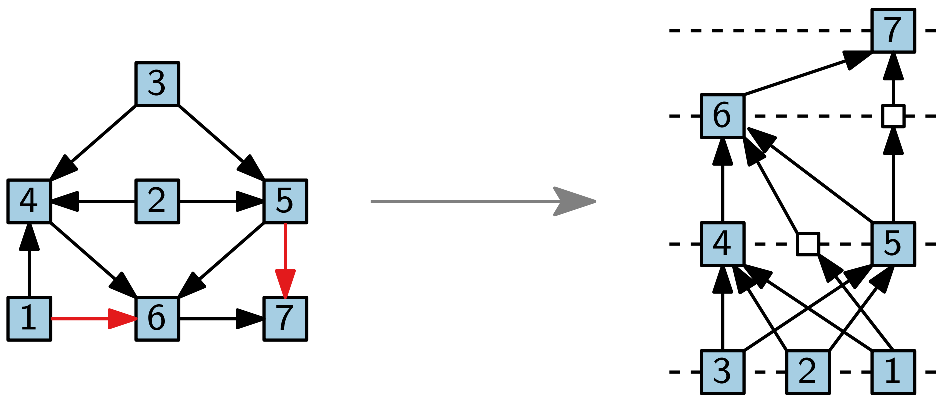
- for each source q
  set $y(q) := 1$



Min number of layers

Algorithm.

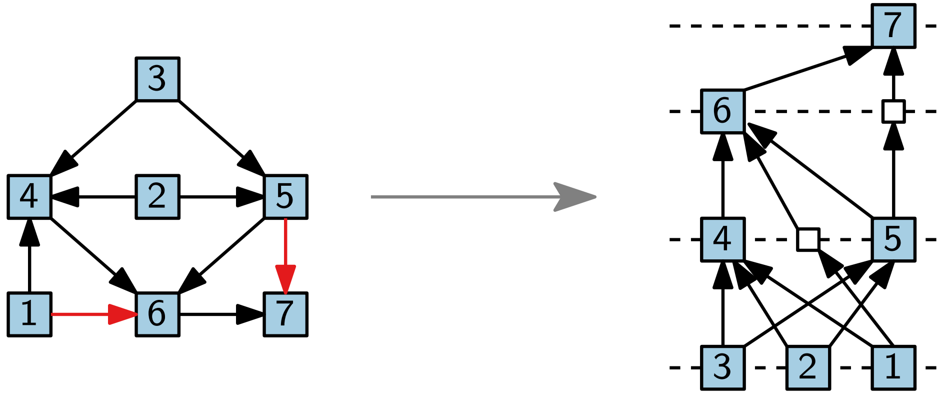
- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



Min number of layers

Algorithm.

- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\left\{y(u) \mid uv \in E\right\} + 1$



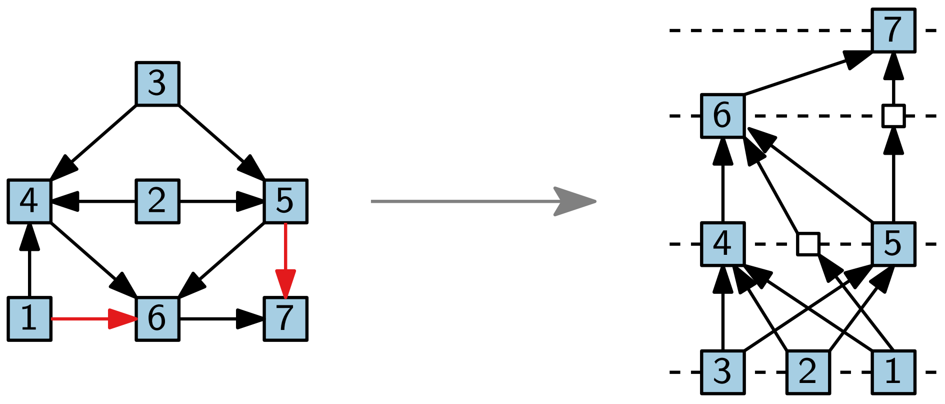
Observation.

- $y(v)$

Min number of layers

Algorithm.

- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



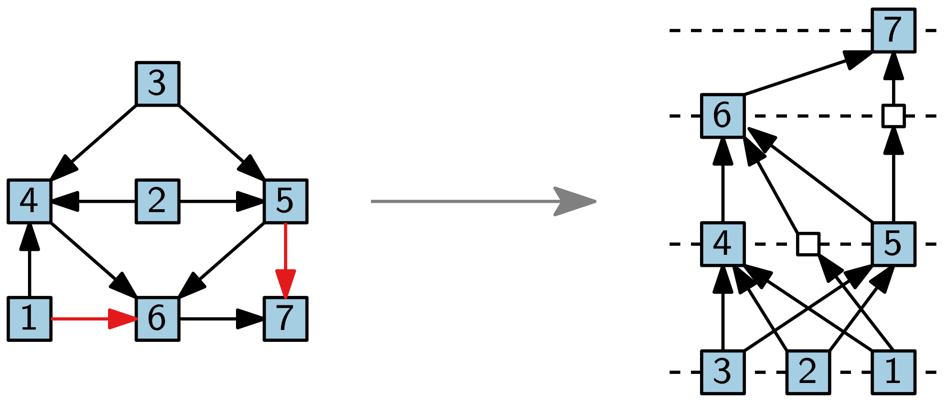
Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.

Min number of layers

Algorithm.

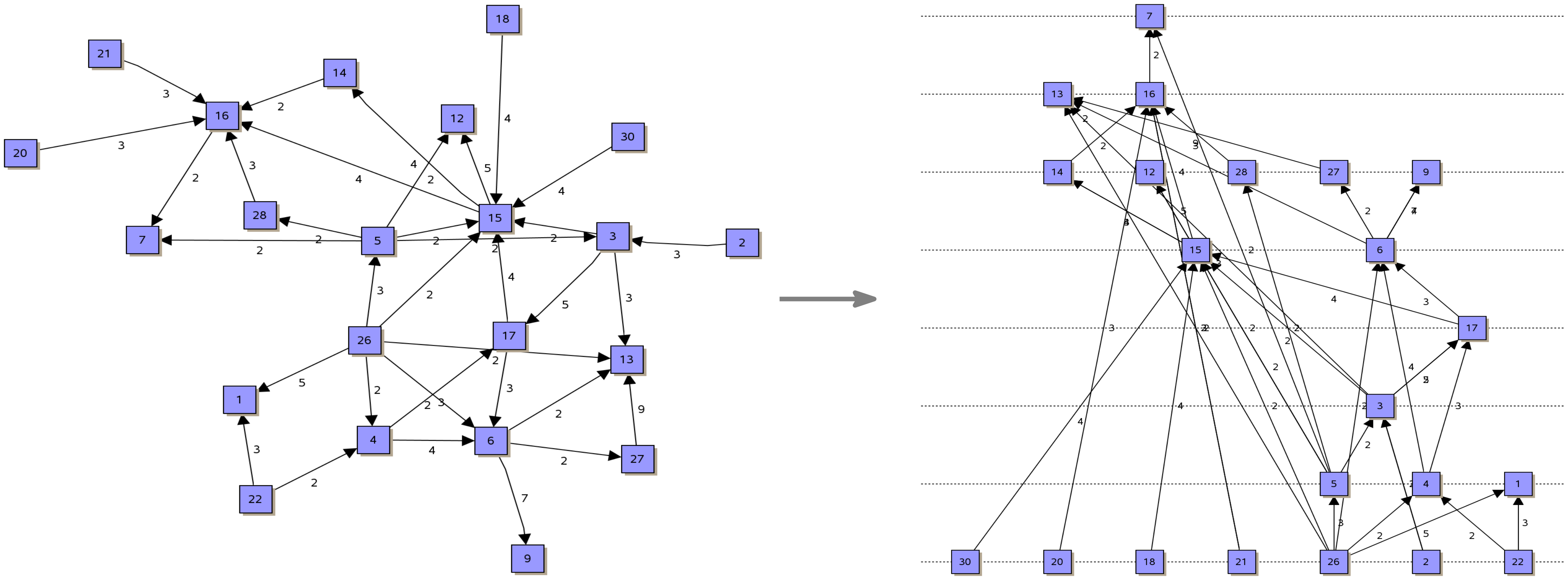
- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



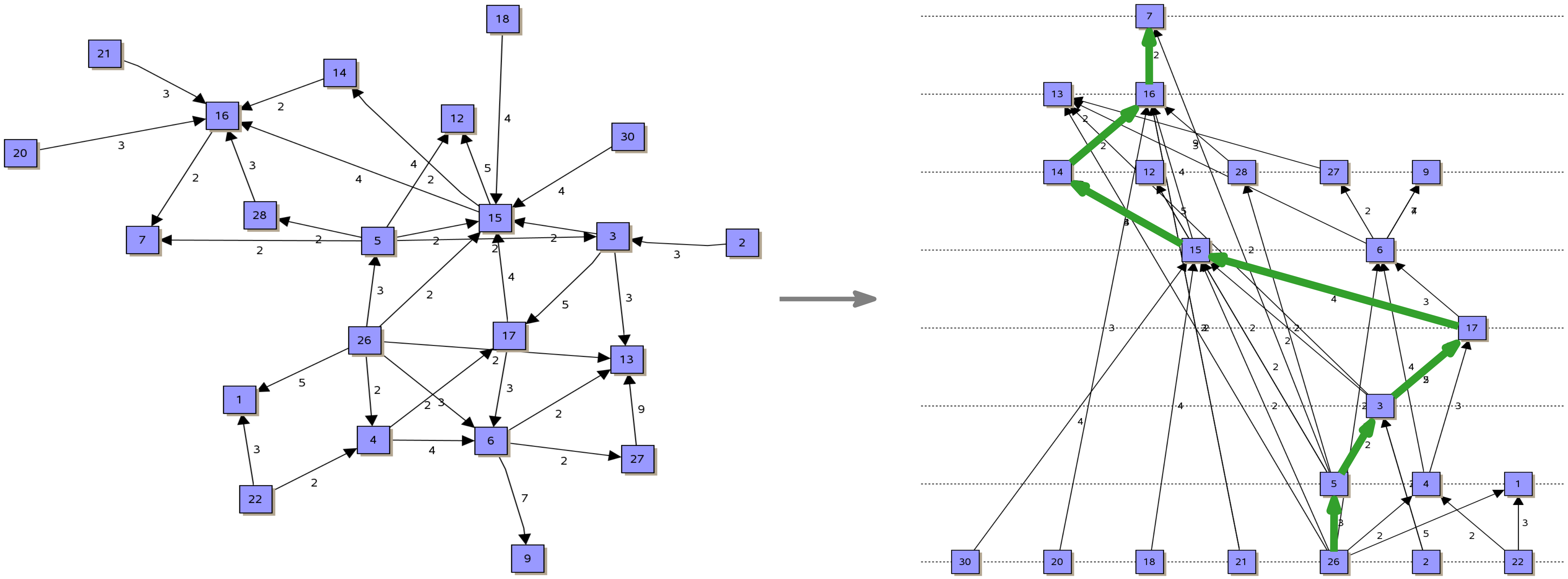
Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.
  ... which is optimal!
- Can be implemented in linear time with recursive algorithm.

Example


21
14
18
16
12
30
20
28
15
7
5
3
2
26
17
13
1
4
6
27
22
9
7
13
16
14
12
28
27
9
15
6
17
3
5
4
1
30
20
18
21
26
2
22

Example



Total edge length – ILP

Can be formulated as an integer linear program:

$$\begin{array}{rll} \text{min} & \sum_{(u,v)\in E}(y(v)-y(u)) & \\ \text{subject to} & y(v)-y(u)\geq 1 & \forall (u,v)\in E \\ & y(v)\geq 1 & \forall v\in V \\ & y(v)\in \mathbb{Z} & \forall v\in V \end{array}$$

Total edge length – ILP

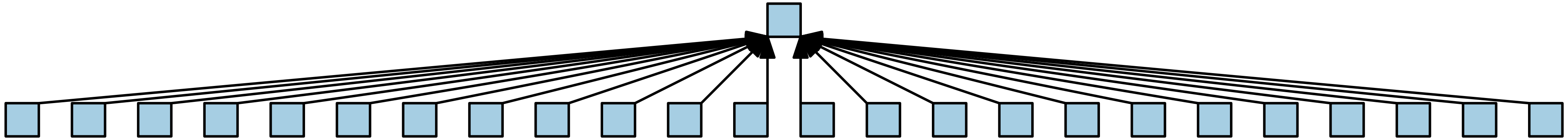
Can be formulated as an integer linear program:

$$\begin{array}{rll} \text{min} & \sum_{(u,v)\in E} (y(v) - y(u)) & \\ \text{subject to} & y(v) - y(u) \geq 1 & \forall (u,v) \in E \\ & y(v) \geq 1 & \forall v \in V \\ & y(v) \in \mathbb{Z} & \forall v \in V \end{array}$$

One can show that:

- Constraint-matrix is **totally unimodular**
  $\Rightarrow$ Solution of the relaxed linear program is integer
- The total edge length can be minimized in polynomial time

Width



Drawings can be very wide.

Narrower layer assignment

Problem: Leveling with a given width.

- Input: acyclic, digraph $G = (V, E)$, width $W > 0$
- Output: Partition the vertex set into a minimum number of layers such that each layer contains at most W elements.

Narrower layer assignment

Problem: Leveling with a given width.

- Input: acyclic, digraph $G = (V, E)$, width $W > 0$
- Output: Partition the vertex set into a minimum number of layers such that each layer contains at most W elements.

Problem: Precedence-Constrained Multi-Processor Scheduling

- Input: n jobs with unit (1) processing time, W identical machines, and a partial ordering $<$ on the jobs.
- Output: Schedule respecting $<$ and having minimum processing time.

Narrower layer assignment

Problem: Leveling with a given width.

- Input: acyclic, digraph $G = (V, E)$, width $W > 0$
- Output: Partition the vertex set into a minimum number of layers such that each layer contains at most W elements.

Problem: Precedence-Constrained Multi-Processor Scheduling

- Input: n jobs with unit (1) processing time, W identical machines, and a partial ordering $<$ on the jobs.
- Output: Schedule respecting $<$ and having minimum processing time.

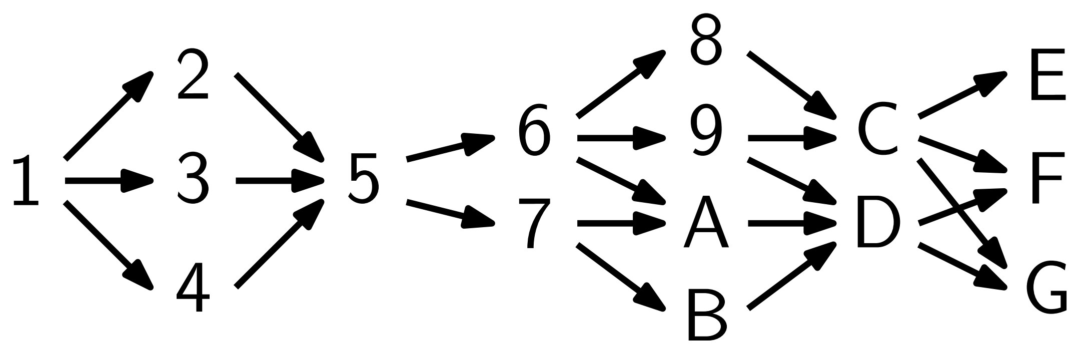
- NP-hard, $(2 - \frac{2}{W})$-Approx., no $(\frac{4}{3} - \varepsilon)$-Approx. $(W \geq 3)$.

Approximating PCMPS

- jobs stored in a list L
  (in any order, e.g., topologically sorted)
- for each time $t = 1, 2, \ldots$ schedule $\leq W$ available jobs
- a job in L is *available* when all its predecessors have been scheduled
- as long as there are free machines and available jobs, take the first available job and assign it to a free machine

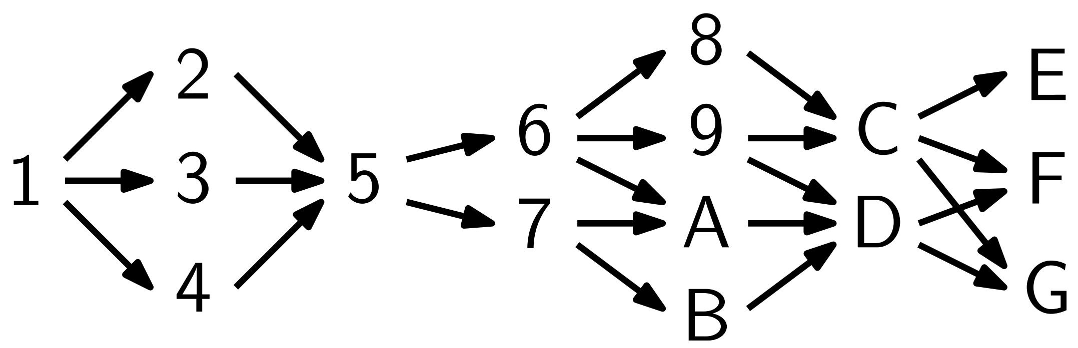
Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



Approximating PCMPS

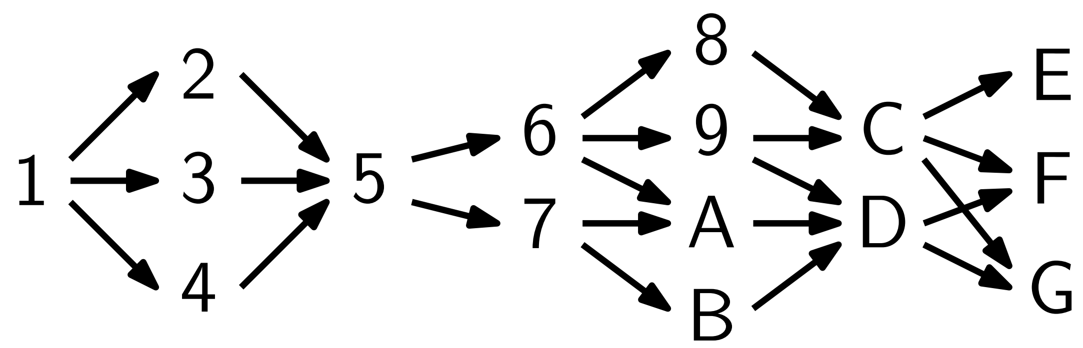
Input: Precedence graph (divided into layers of arbitrary width)



Number of Machines is $W = 2$.

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)

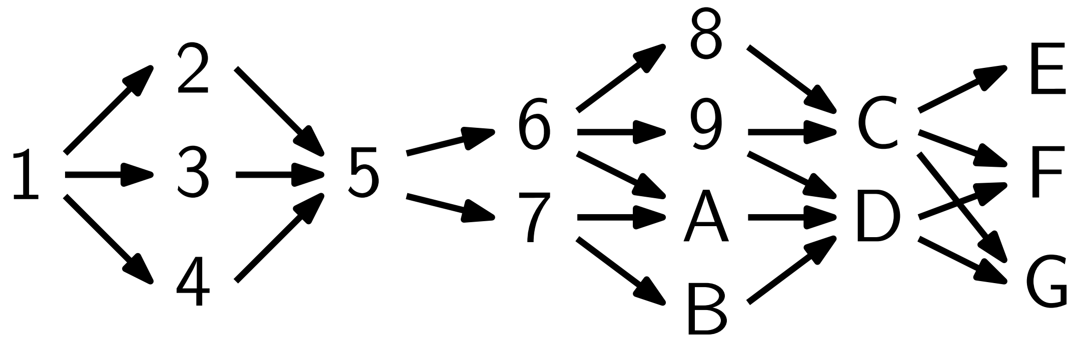


Number of Machines is $W = 2$.

Output: Schedule

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



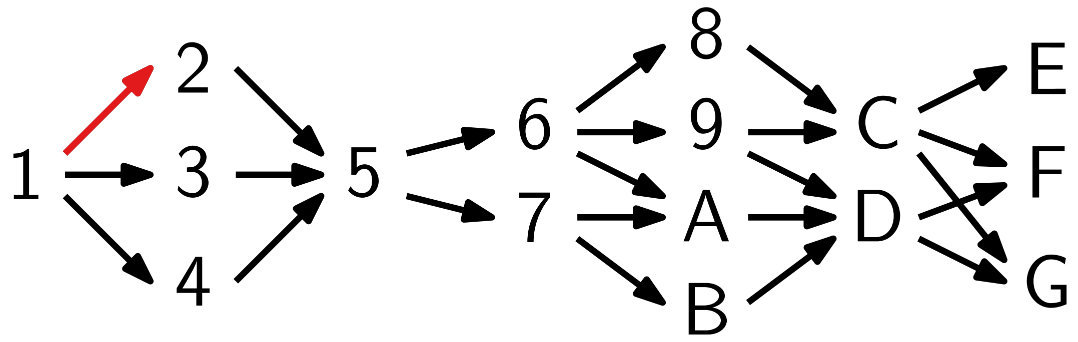
Number of Machines is $W = 2$.

Output: Schedule

M_1										
M_2										
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



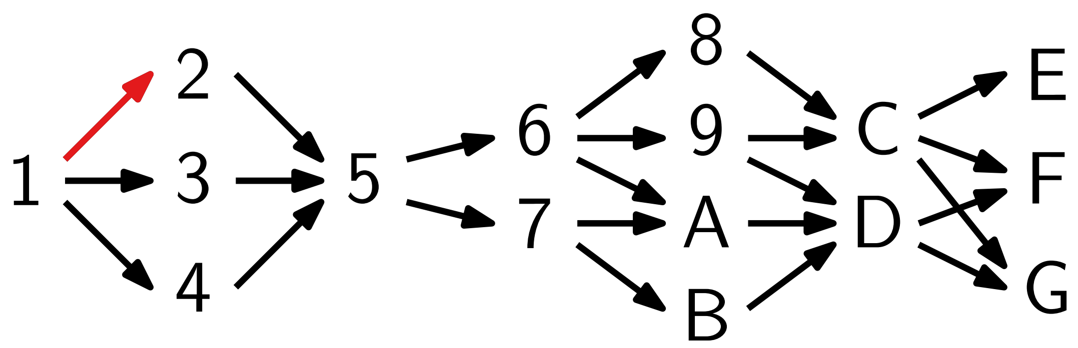
Number of Machines is $W = 2$.

Output: Schedule

M_1	1									
M_2	–									
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



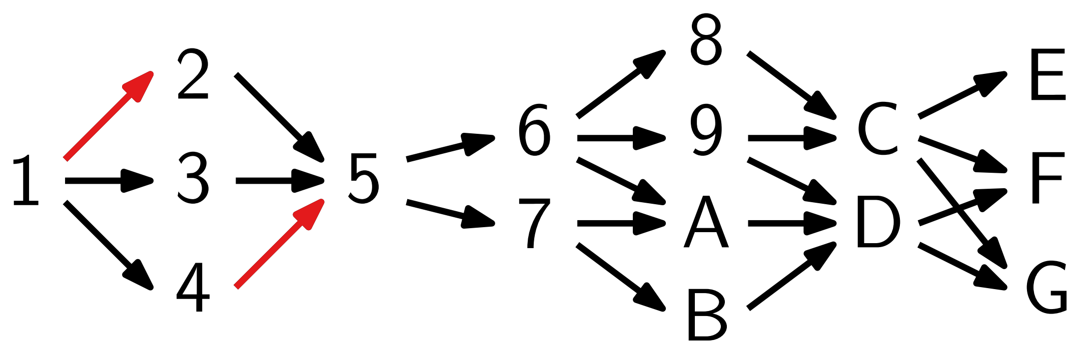
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2								
M_2	–	3								
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



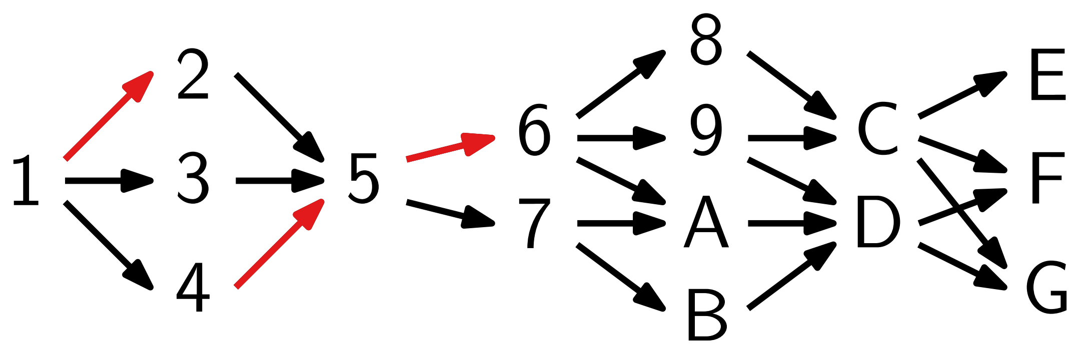
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4							
M_2	–	3	–							
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



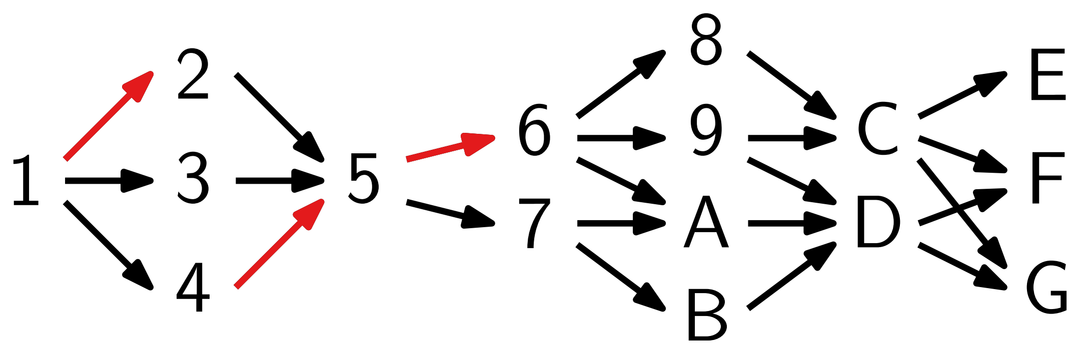
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5						
M_2	–	3	–	–						
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



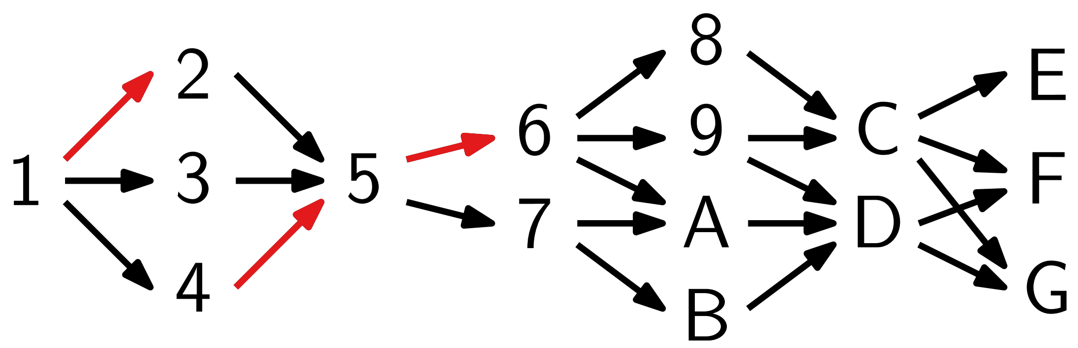
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6					
M_2	–	3	–	–	7					
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



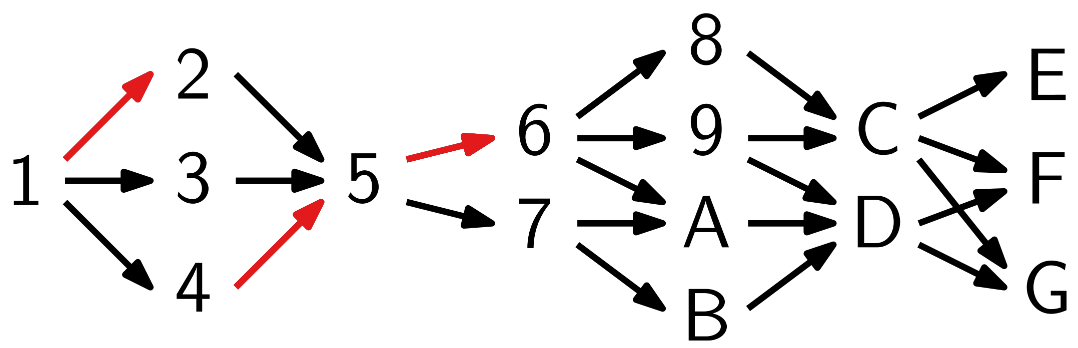
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8				
M_2	–	3	–	–	7	9				
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



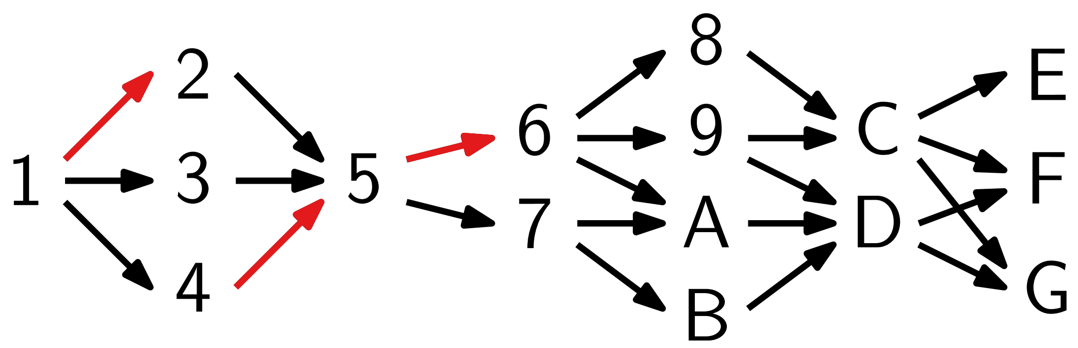
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A			
M_2	–	3	–	–	7	9	B			
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



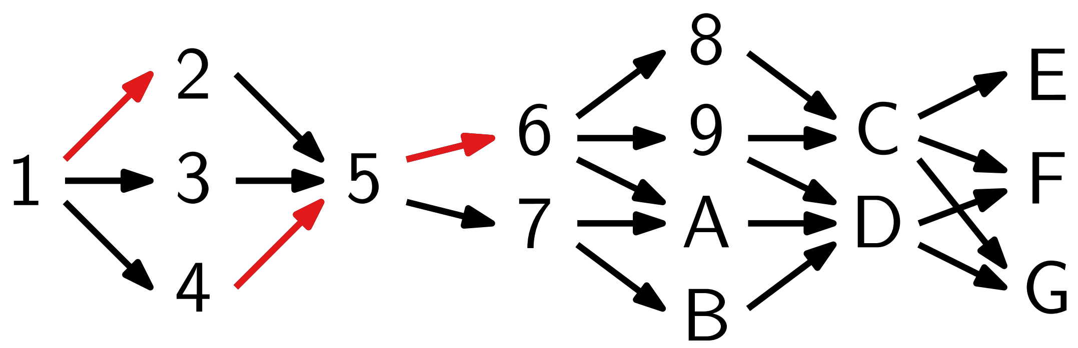
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C		
M_2	–	3	–	–	7	9	B	D		
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



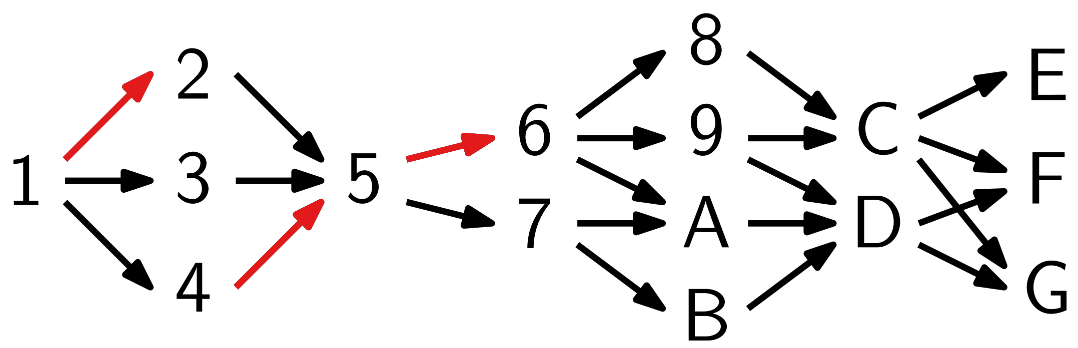
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	
M_2	–	3	–	–	7	9	B	D	F	
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



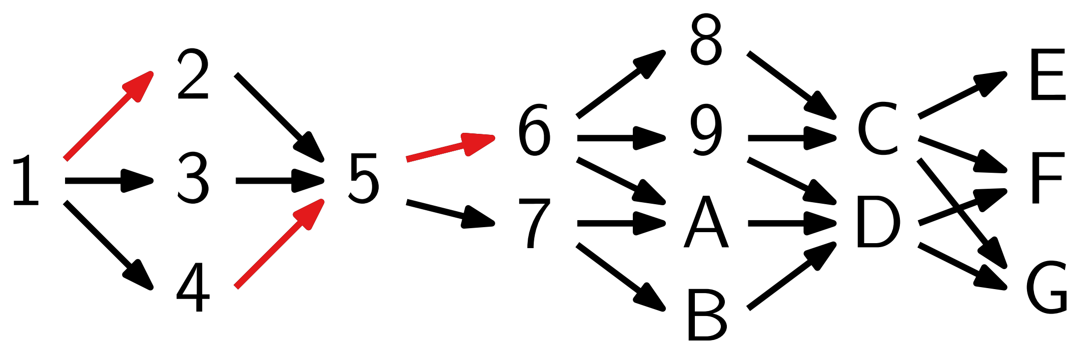
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



Number of Machines is $W = 2$.

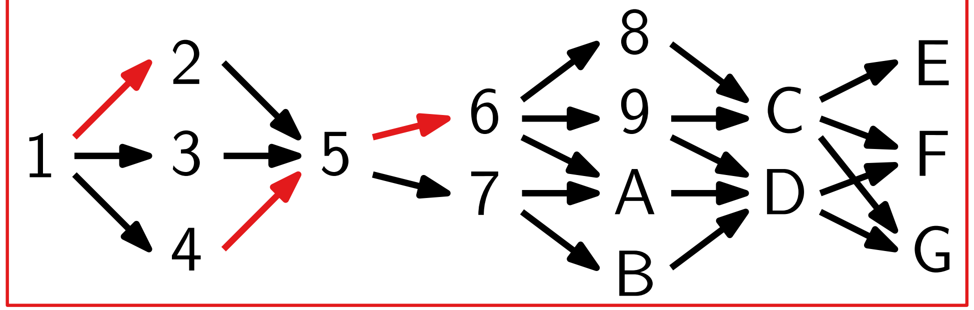
Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

Question: Good approximation factor?

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



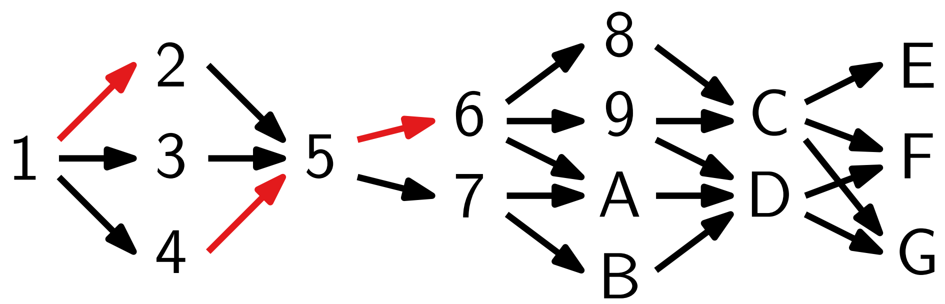
Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

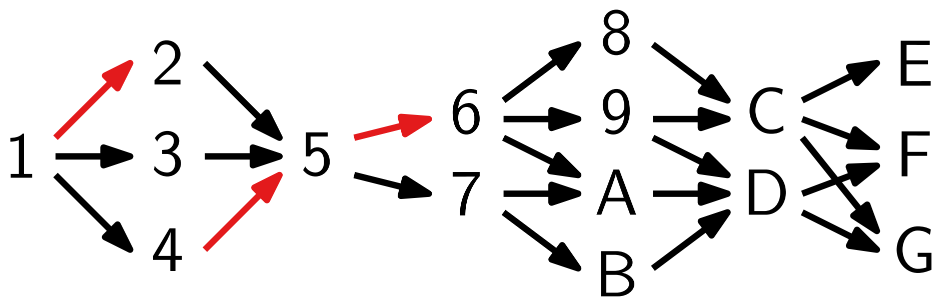
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

OPT $\geq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

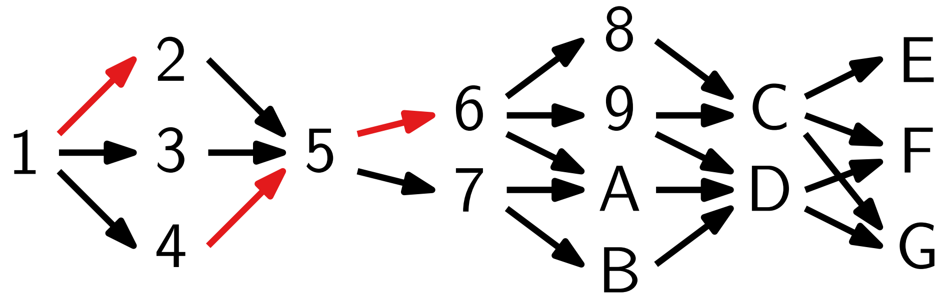
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

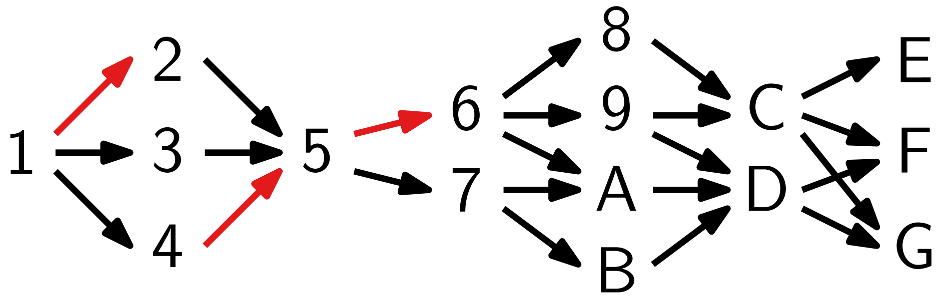
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

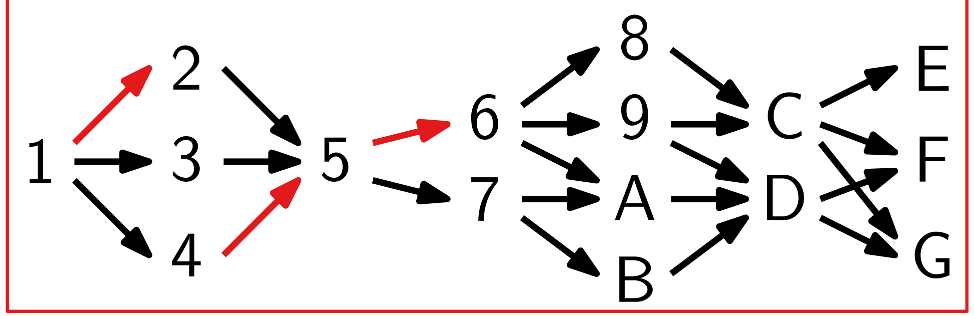
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

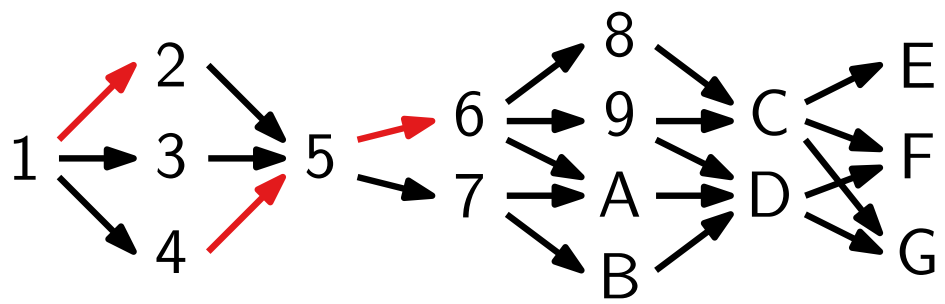
„The art of the lower bound"

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

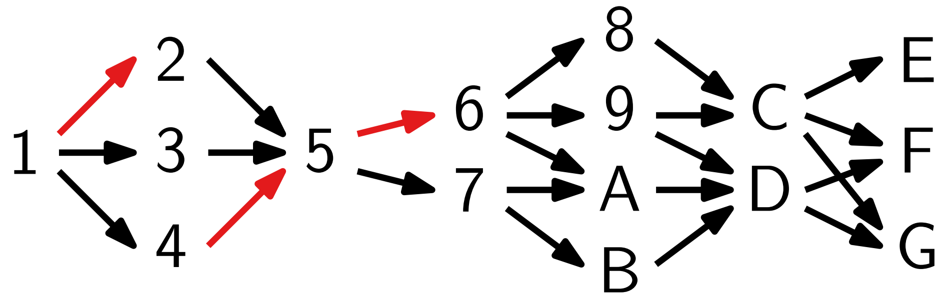
OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

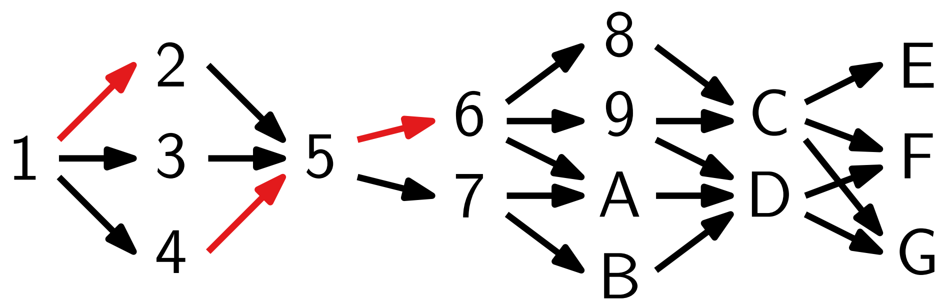
OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

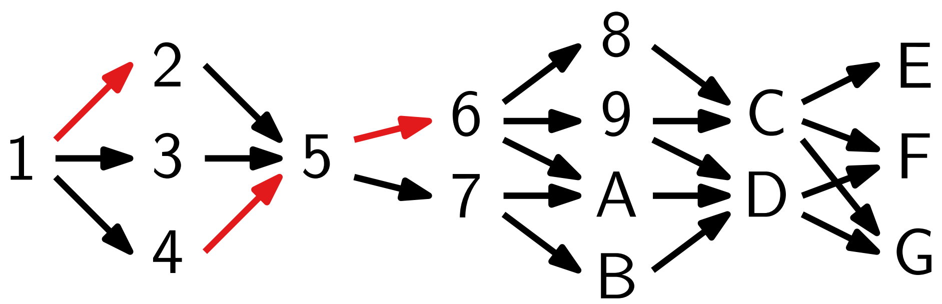
Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

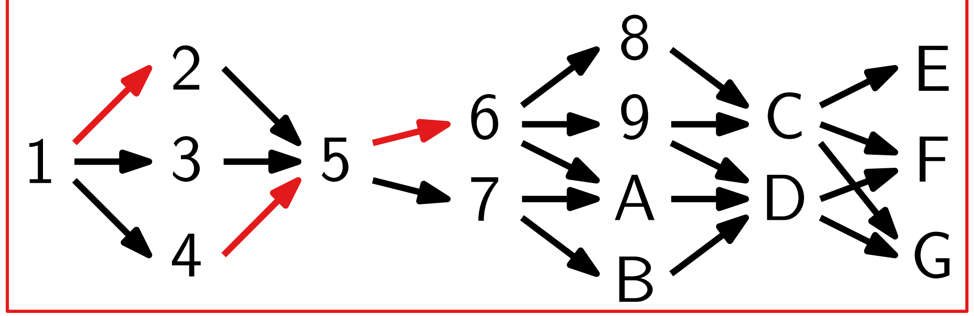
Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

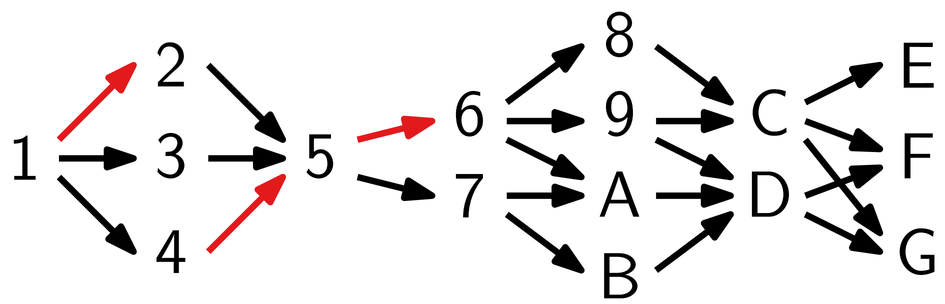
Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

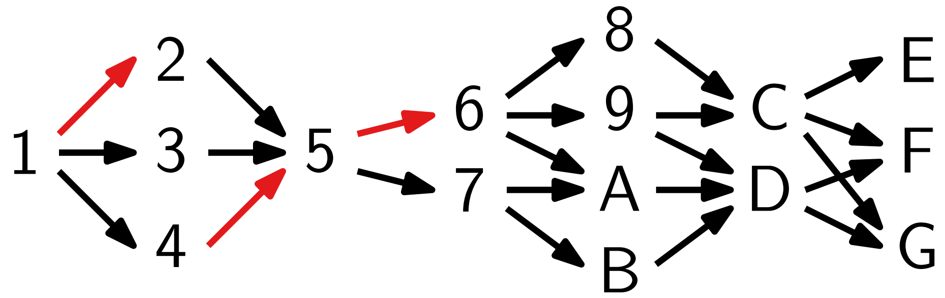
Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

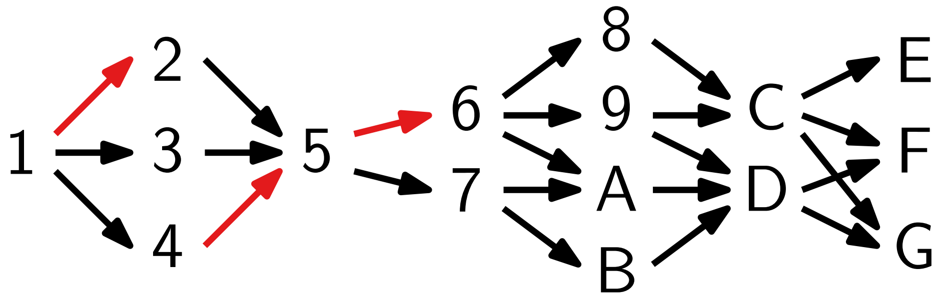
Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

$\mathrm{OPT} \geq \lceil n/2 \rceil$ and $\mathrm{OPT} \geq \ell :=$ Number of layers of $G_<$

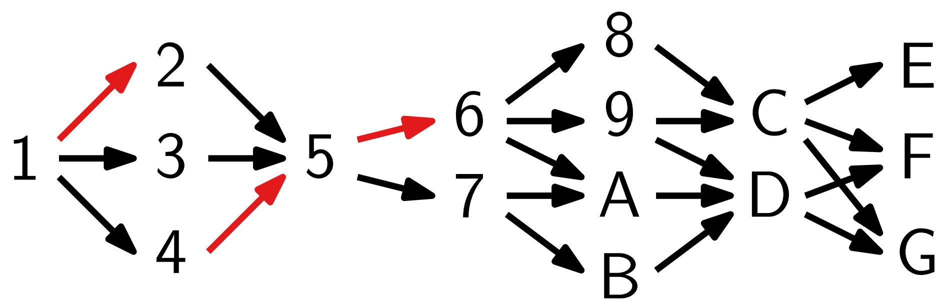
Goal: measure the quality of our algorithm using the lower bounds

Bound. $\mathrm{ALG} \leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot \mathrm{OPT}$

insertion of pauses (–) in the schedule
(except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

$\mathrm{OPT} \geq \lceil n/2 \rceil$ and $\mathrm{OPT} \geq \ell :=$ Number of layers of $G_<$

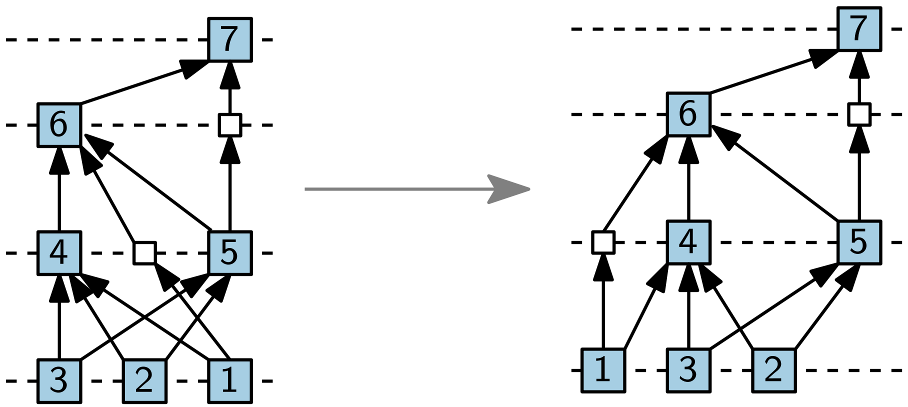
Goal: measure the quality of our algorithm using the lower bounds

$\leq (2 - 1/W) \cdot \mathrm{OPT}$ in general case

Bound. $\mathrm{ALG} \leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot \mathrm{OPT}$

insertion of pauses (–) in the schedule
(except the last) maps to layers of $G_<$

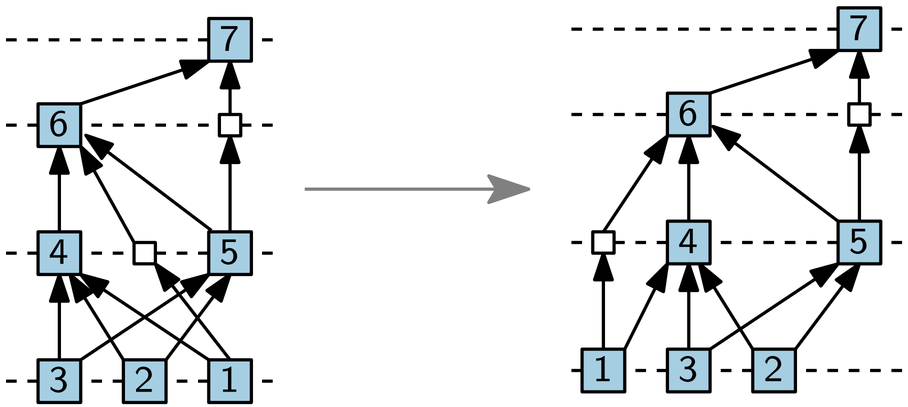
Step 3: Crossing minimization



Problem.

- Input: Graph G, layering $y\colon V \to \{1, \ldots, |V|\}$
- Output: (Re-)ordering of vertices in each layer so that the number of crossings in minimized.

Step 3: Crossing minimization



Problem.

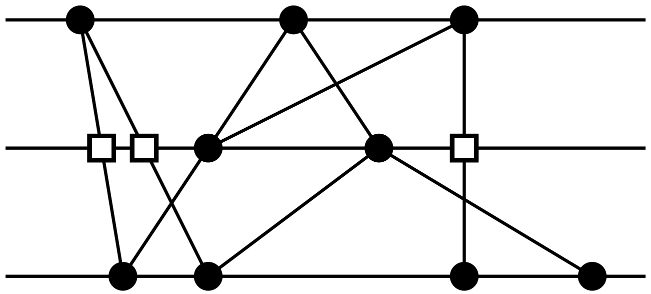
- Input: Graph G, layering $y \colon V \to \{1, \ldots, |V|\}$
- Output: (Re-)ordering of vertices in each layer so that the number of crossings in minimized.

- NP-hard, even for 2 layers [Garey & Johnson '83]
- hardly any approaches optimize over multiple layers :(

Iterative crossing reduction – idea

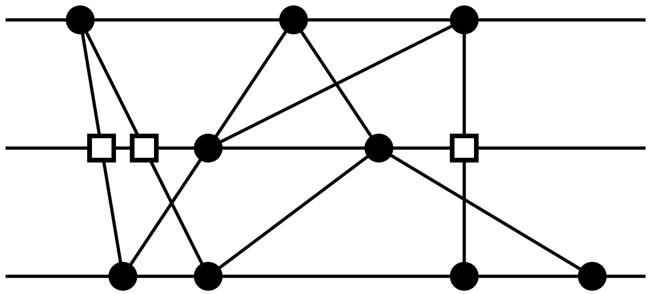
Observation.

The number of crossings only depends on permutations of adjacent layers.



Iterative crossing reduction – idea

Observation.
The number of crossings only depends on permutations of adjacent layers.



- Add dummy-vertices for edges connecting “far” layers.
- Consider adjacent layers $(L_1, L_2), (L_2, L_3), \ldots$ bottom-to-top.
- Minimize crossings by permuting L_{i+1} while keeping L_i fixed.

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

(3) minimize crossings by permuting L_{i+1} and keeping L_i fixed

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

(3) minimize crossings by permuting L_{i+1} and keeping L_i fixed

(4) repeat steps (2)–(3) in the reverse order (starting from L_h)

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

(3) minimize crossings by permuting L_{i+1} and keeping L_i fixed

(4) repeat steps (2)–(3) in the reverse order (starting from L_h)

(5) repeat steps (2)–(4) until no further improvement is achieved

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

(3) minimize crossings by permuting L_{i+1} and keeping L_i fixed

(4) repeat steps (2)–(3) in the reverse order (starting from L_h)

(5) repeat steps (2)–(4) until no further improvement is achieved

(6) repeat steps (1)–(5) with different starting permutations

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

(3) minimize crossings by permuting L_{i+1} and keeping L_i fixed

(4) repeat steps (2)–(3) in the reverse order (starting from L_h)

(5) repeat steps (2)–(4) until no further improvement is achieved

(6) repeat steps (1)–(5) with different starting permutations

Iterative crossing reduction – algorithm

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

(3) minimize crossings by permuting L_{i+1} and keeping L_i fixed *one-sided crossing minimization*

(4) repeat steps (2)–(3) in the reverse order (starting from L_h)

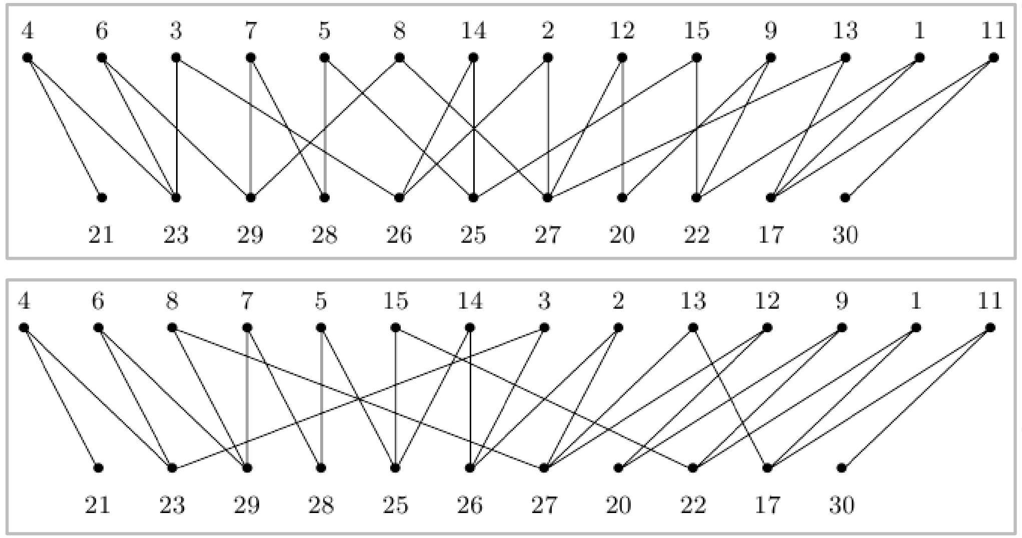
(5) repeat steps (2)–(4) until no further improvement is achieved

(6) repeat steps (1)–(5) with different starting permutations

One-sided crossing minimization

Problem.

- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation π_1 on L_1
- Output: permutation π_2 of L_2 minimizing the number of edge crossings.



One-sided crossing minimization

Problem.

- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation π_1 on L_1
- Output: permutation π_2 of L_2 minimizing the number of edge crossings.

One-sided crossing minimization is NP-hard.
[Eades & Whitesides '94]

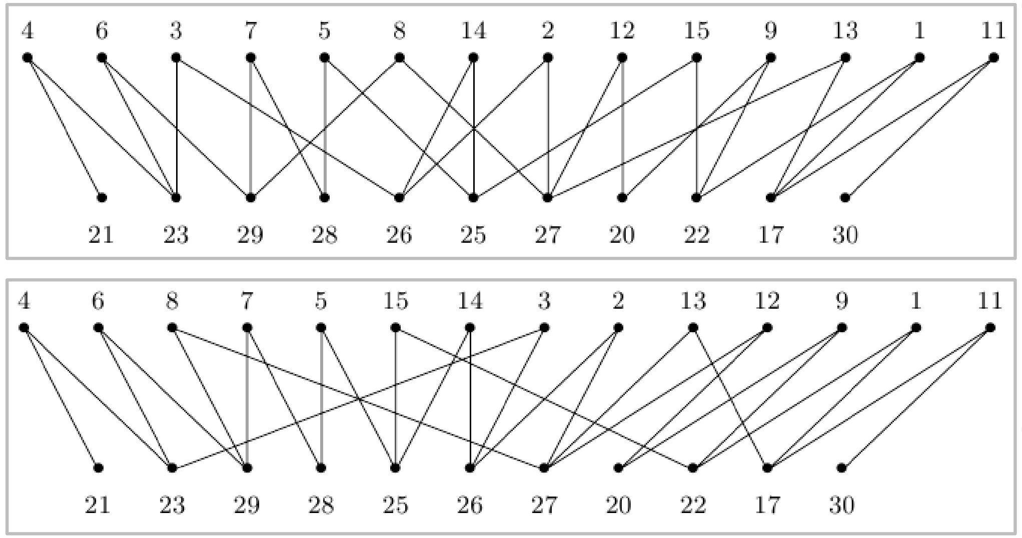


Abb. aus [Kaufmann und Wagner: Drawing Graphs] (c) Springer-Verlag

One-sided crossing minimization

Problem.

- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation π_1 on L_1
- Output: permutation π_2 of L_2 minimizing the number of edge crossings.

One-sided crossing minimization is NP-hard.
[Eades & Whitesides '94]

Algorithms.

- barycenter heuristic
- median heuristic
- Greedy-Switch
- ILP
- ...

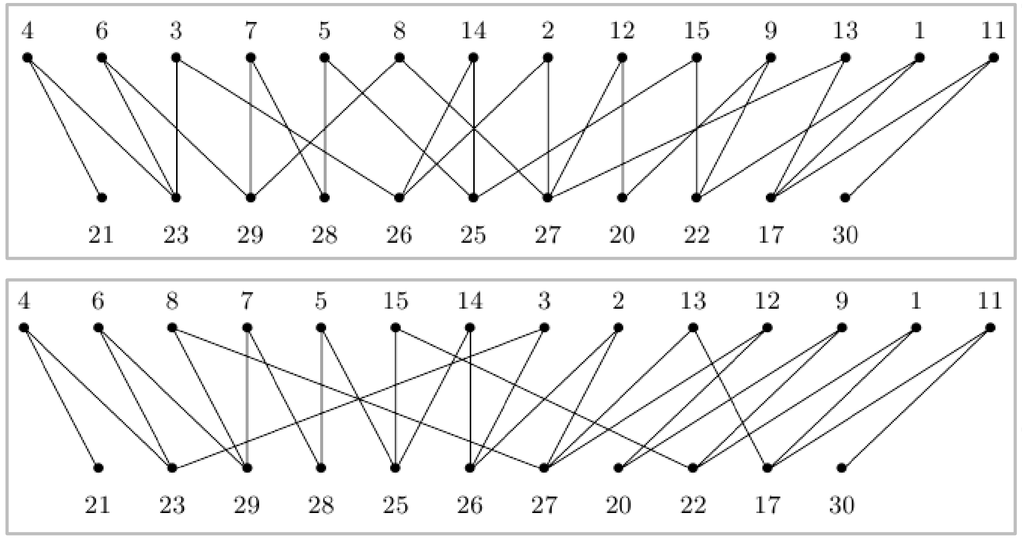


Abb. aus [Kaufmann und Wagner: Drawing Graphs]
(c) Springer-Verlag

Barycentre heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbours

Barycentre heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbours
- The barycentre of u is the average x-coordinate of the neighbours of u in layer L_1 $[x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

Barycentre heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbours
- The barycentre of u is the average x-coordinate of the neighbours of u in layer L_1 $[x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- Vertices with the same barycentre of are offset by a small δ.

Barycentre heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbours
- The barycentre of u is the average x-coordinate of the neighbours of u in layer L_1 $[x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- Vertices with the same barycentre of are offset by a small δ.
- linear runtime
- relatively good results
- optimal if no crossings are required
- $O(\sqrt{n})$-approximation factor

Barycentre heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbours
- The barycentre of u is the average x-coordinate of the neighbours of u in layer L_1 $[x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- Vertices with the same barycentre of are offset by a small δ.
- linear runtime
- relatively good results
- optimal if no crossings are required ← *exercise!*
- $O(\sqrt{n})$-approximation factor

Barycentre heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbours
- The barycentre of u is the average x-coordinate of the neighbours of u in layer L_1 $\quad [x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- Vertices with the same barycentre of are offset by a small δ.

- linear runtime
- relatively good results
- optimal if no crossings are required ← *exercise!*
- $O(\sqrt{n})$-approximation factor

Worst case?

Barycentre heuristic

[Sugiyama et al. '81]

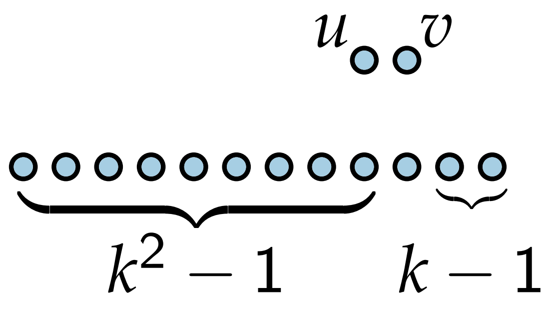
- Intuition: few intersections occur when vertices are close to their neighbours
- The barycentre of u is the average x-coordinate of the neighbours of u in layer L_1 $[x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- Vertices with the same barycentre of are offset by a small δ.

- linear runtime
- relatively good results
- optimal if no crossings are required ← *exercise!*
- $O(\sqrt{n})$-approximation factor

Worst case?



Barycentre heuristic

[Sugiyama et al. '81]

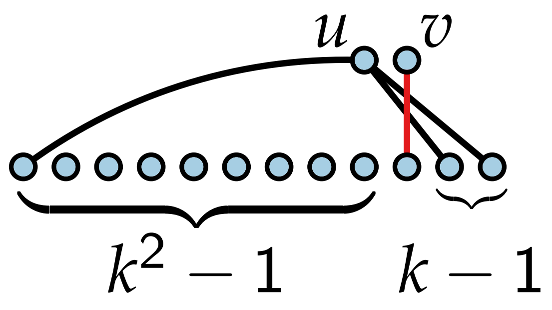
- Intuition: few intersections occur when vertices are close to their neighbours
- The barycentre of u is the average x-coordinate of the neighbours of u in layer L_1 $[x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- Vertices with the same barycentre of are offset by a small δ.

- linear runtime
- relatively good results
- optimal if no crossings are required ← *exercise!*
- $O(\sqrt{n})$-approximation factor

Worst case?



Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
- $$x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$$
- move vertices u und v by small δ, when $x_2(u) = x_2(v)$

Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
- $$x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$$
- move vertices u und v by small δ, when $x_2(u) = x_2(v)$

- linear runtime
- relatively good results
- optimal, if no crossings are required
- 3-approximation factor

Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
- $$x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$$
- move vertices u und v by small δ, when $x_2(u) = x_2(v)$

- linear runtime
- relatively good results
- optimal, if no crossings are required
- 3-approximation factor
  proof in [GD Ch 11]

Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
- $$x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$$
- move vertices u und v by small δ, when $x_2(u) = x_2(v)$

- linear runtime
- relatively good results
- optimal, if no crossings are required
- 3-approximation factor
  proof in [GD Ch 11]

Worst case?

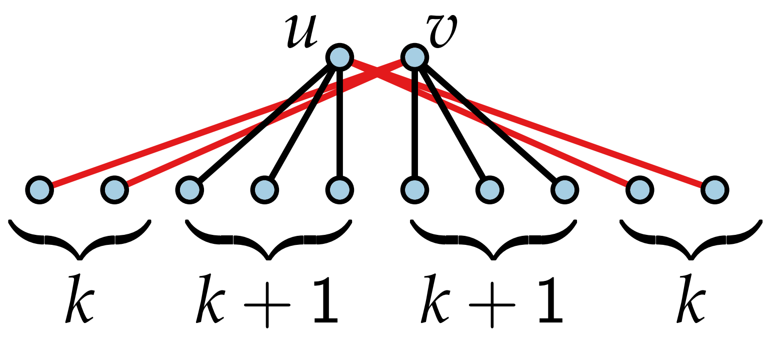
Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
- $$x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$$
- move vertices u und v by small δ, when $x_2(u) = x_2(v)$

- linear runtime
- relatively good results
- optimal, if no crossings are required
- 3-approximation factor
  proof in [GD Ch 11]

Worst case?



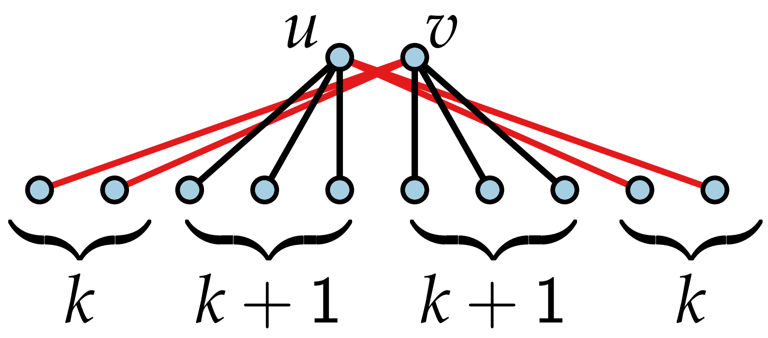
Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
- $$x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$$
- move vertices u und v by small δ, when $x_2(u) = x_2(v)$

- linear runtime
- relatively good results
- optimal, if no crossings are required
- 3-approximation factor
  proof in [GD Ch 11]

Worst case?



$2k(k+1) + k^2$ vs. $(k+1)^2$

Greedy-switch heuristic

- iteratively swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- suitable as post-processing for other heuristics

Greedy-switch heuristic

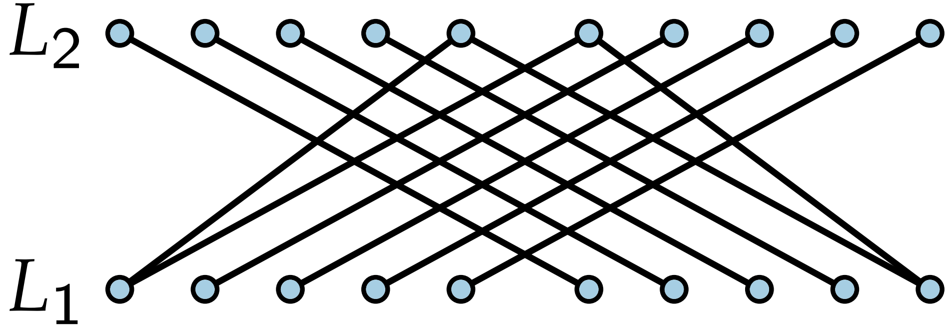
- iteratively swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- suitable as post-processing for other heuristics

Worst case?

Greedy-switch heuristic

- iterativey swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- suitable as post-processing for other heuristics

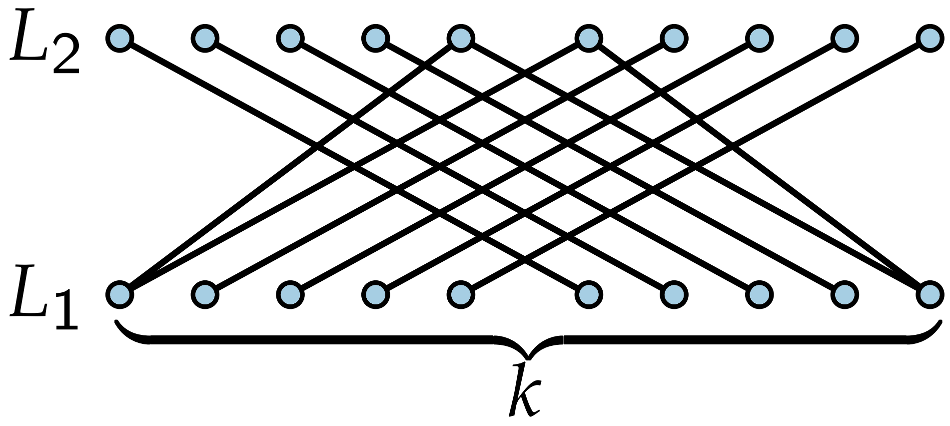
Worst case?



Greedy-switch heuristic

- iteratively swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- suitable as post-processing for other heuristics

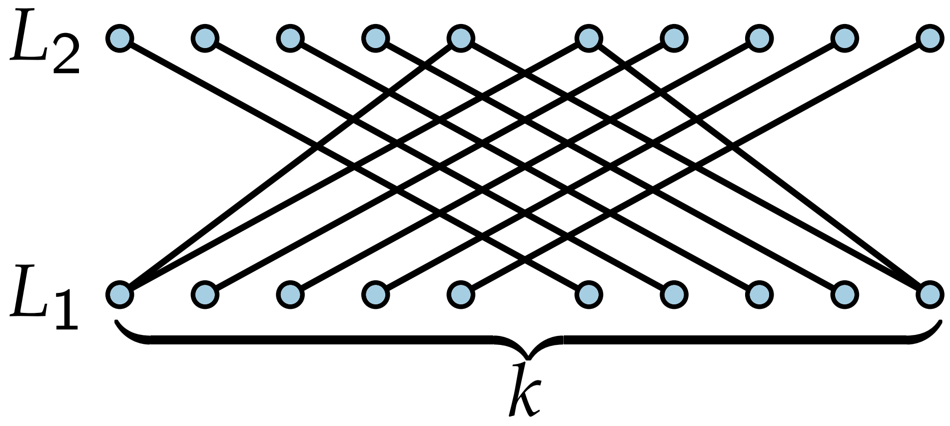
Worst case?

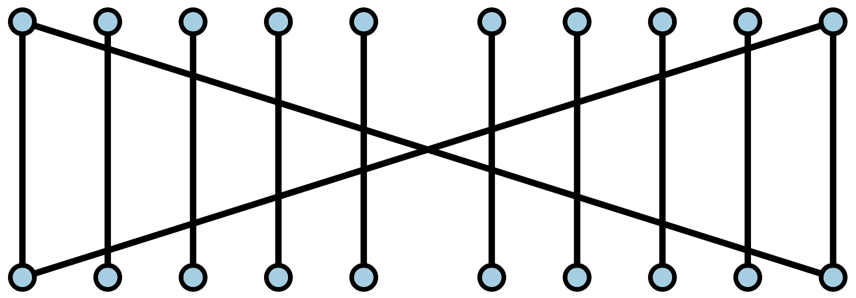


Greedy-switch heuristic

- iteratively swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- suitable as post-processing for other heuristics

Worst case?

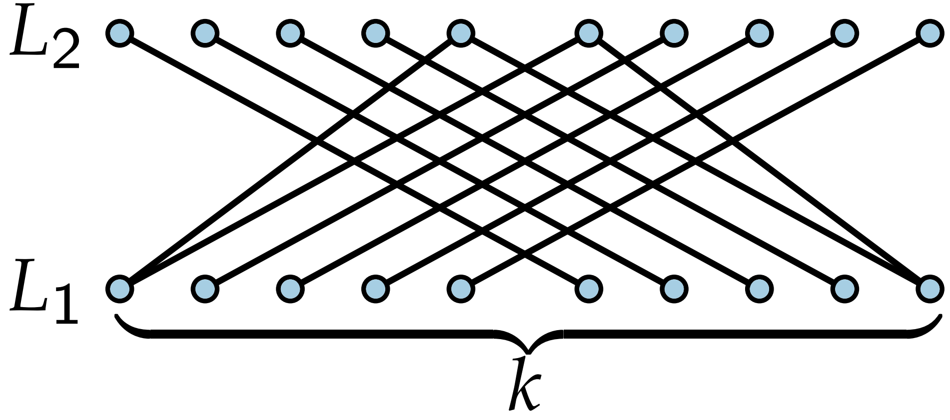




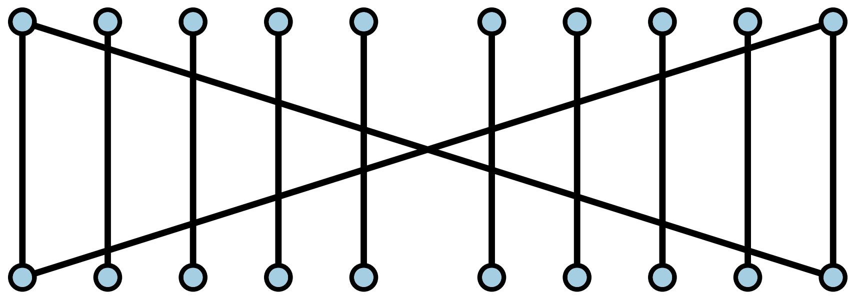
Greedy-switch heuristic

- iteratively swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- suitable as post-processing for other heuristics

Worst case?



$\approx k^2/4$

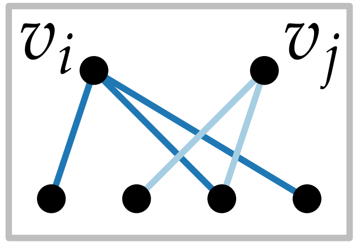


$\approx 2k$

Integer linear program

[Jünger & Mutzel, '97]

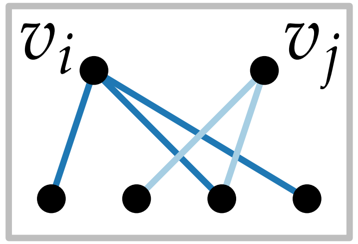
- Constant $c_{ij} :=$ # crossings between edges incident to v_i or v_j when $\pi_2(v_i) < \pi_2(v_j)$



Integer linear program

[Jünger & Mutzel, '97]

- Constant $c_{ij} :=$ # crossings between edges incident to v_i or v_j when $\pi_2(v_i) < \pi_2(v_j)$
- Variable x_{ij} for each $1 \leq i < j \leq n_2 := |L_2|$

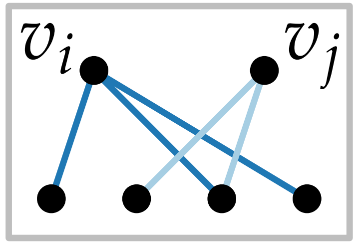


$$x_{ij} = \begin{cases} 1 & \text{when } \pi_2(v_i) < \pi_2(v_j) \\ 0 & \text{otherwise} \end{cases}$$

Integer linear program

[Jünger & Mutzel, '97]

- Constant $c_{ij} :=$ # crossings between edges incident to v_i or v_j when $\pi_2(v_i) < \pi_2(v_j)$
- Variable x_{ij} for each $1 \leq i < j \leq n_2 := |L_2|$



$$x_{ij} = \begin{cases} 1 & \text{when } \pi_2(v_i) < \pi_2(v_j) \\ 0 & \text{otherwise} \end{cases}$$

- The number of crossings of a permutations π_2

$$\text{cross}(\pi_2) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij} + \underbrace{\sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} c_{ji}}_{\text{constant}}$$

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

- Transitivity constraints:

$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \qquad \text{for } 1 \leq i < j < k \leq n_2$$

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

- Transitivity constraints:

$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \qquad \text{for } 1 \leq i < j < k \leq n_2$$

i.e., if $x_{ij} = 1$ and $x_{jk} = 1$, then $x_{ik} = 1$

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

- Transitivity constraints:

$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \qquad \text{for } 1 \leq i < j < k \leq n_2$$

i.e., if $x_{ij} = \overset{1}{0}$ and $x_{jk} = \overset{1}{0}$, then $x_{ik} = \overset{1}{0}$

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

- Transitivity constraints:

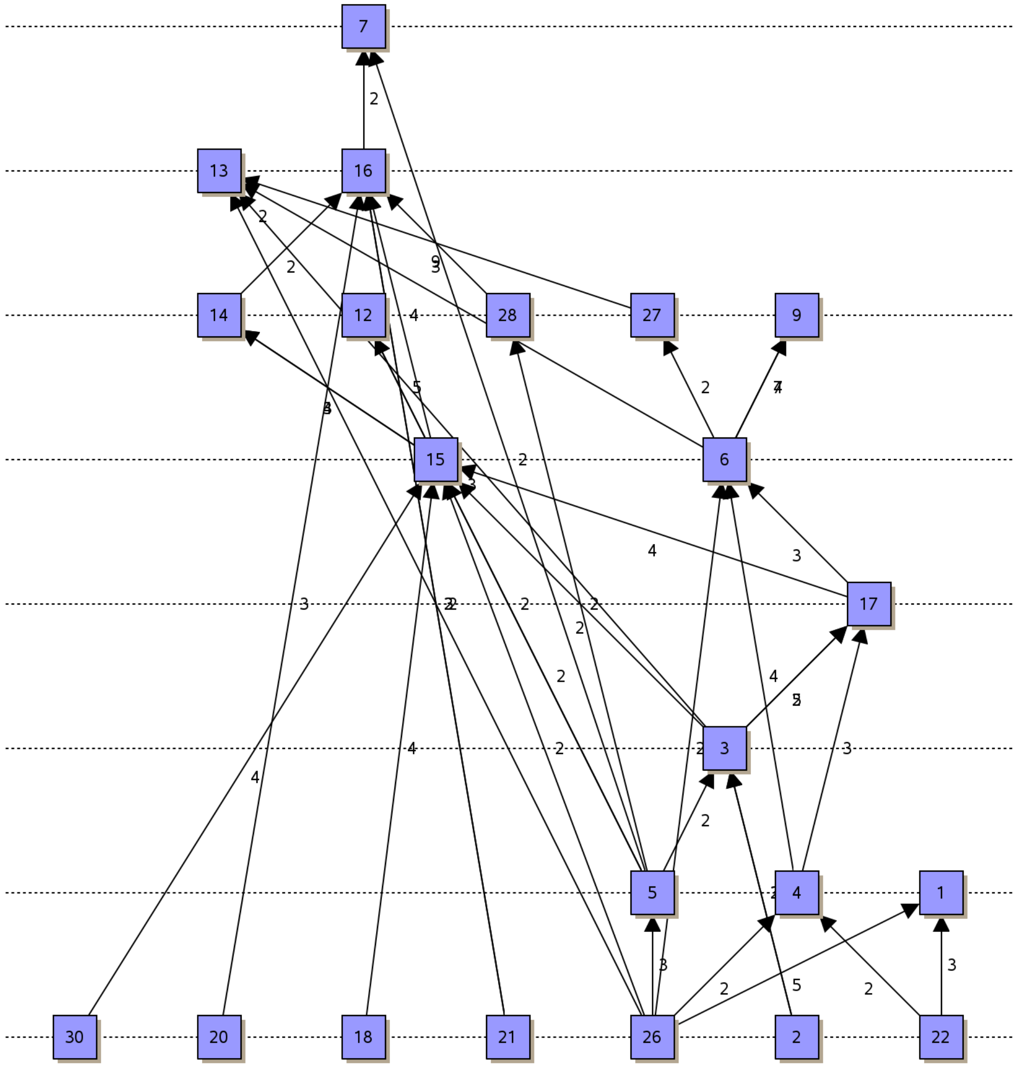
$$0 \le x_{ij} + x_{jk} - x_{ik} \le 1 \qquad \text{for } 1 \le i < j < k \le n_2$$

i.e., if $x_{ij} = \overset{1}{\underset{0}{}}$ and $x_{jk} = \overset{1}{\underset{0}{}}$, then $x_{ik} = \overset{1}{\underset{0}{}}$

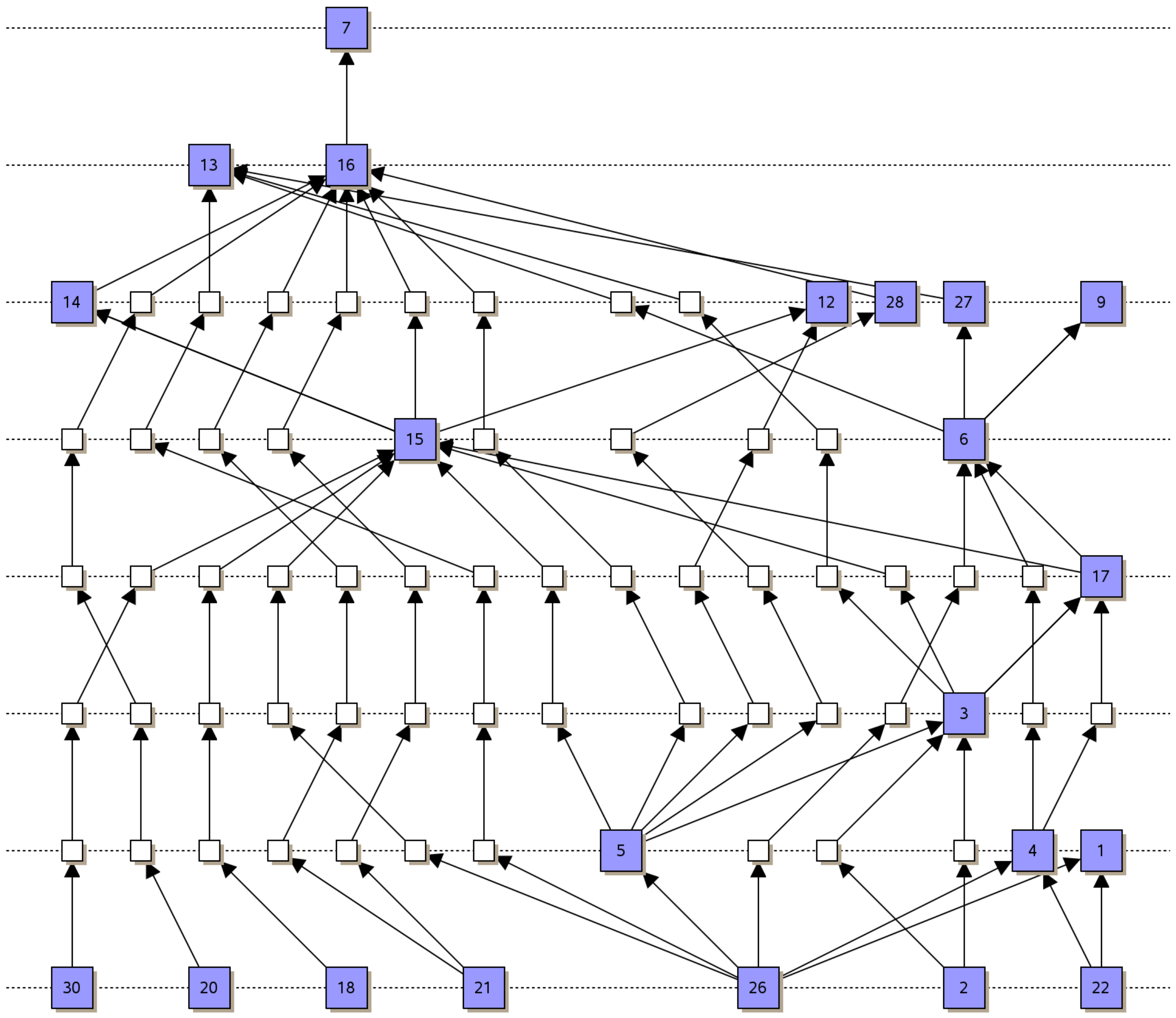
Properties.

- branch-and-cut technique for DAGs of limited size
- useful for graphs of small to medium size
- finds optimal solution
- solution in polynomial time is not guaranteed

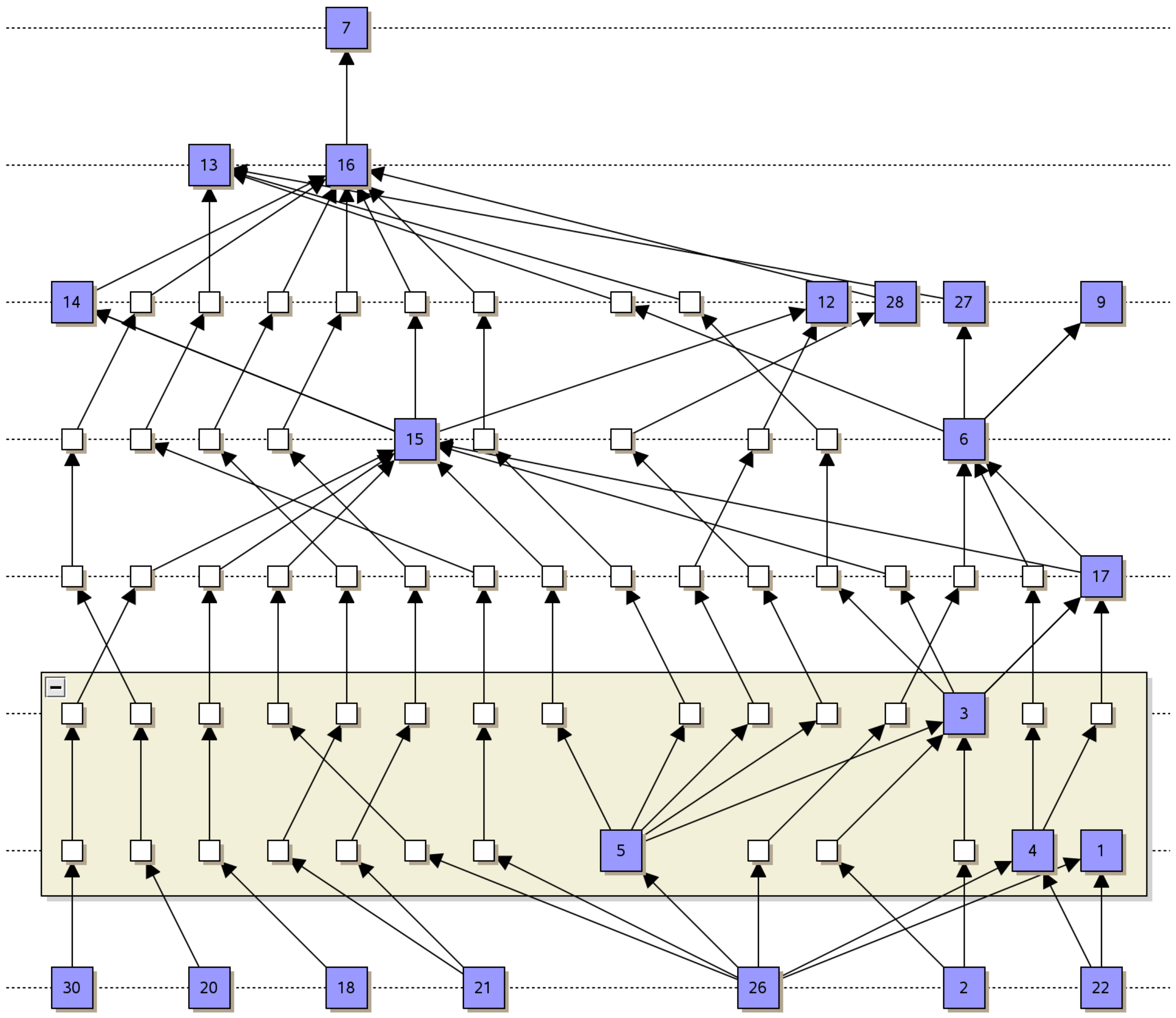
Iterations on example



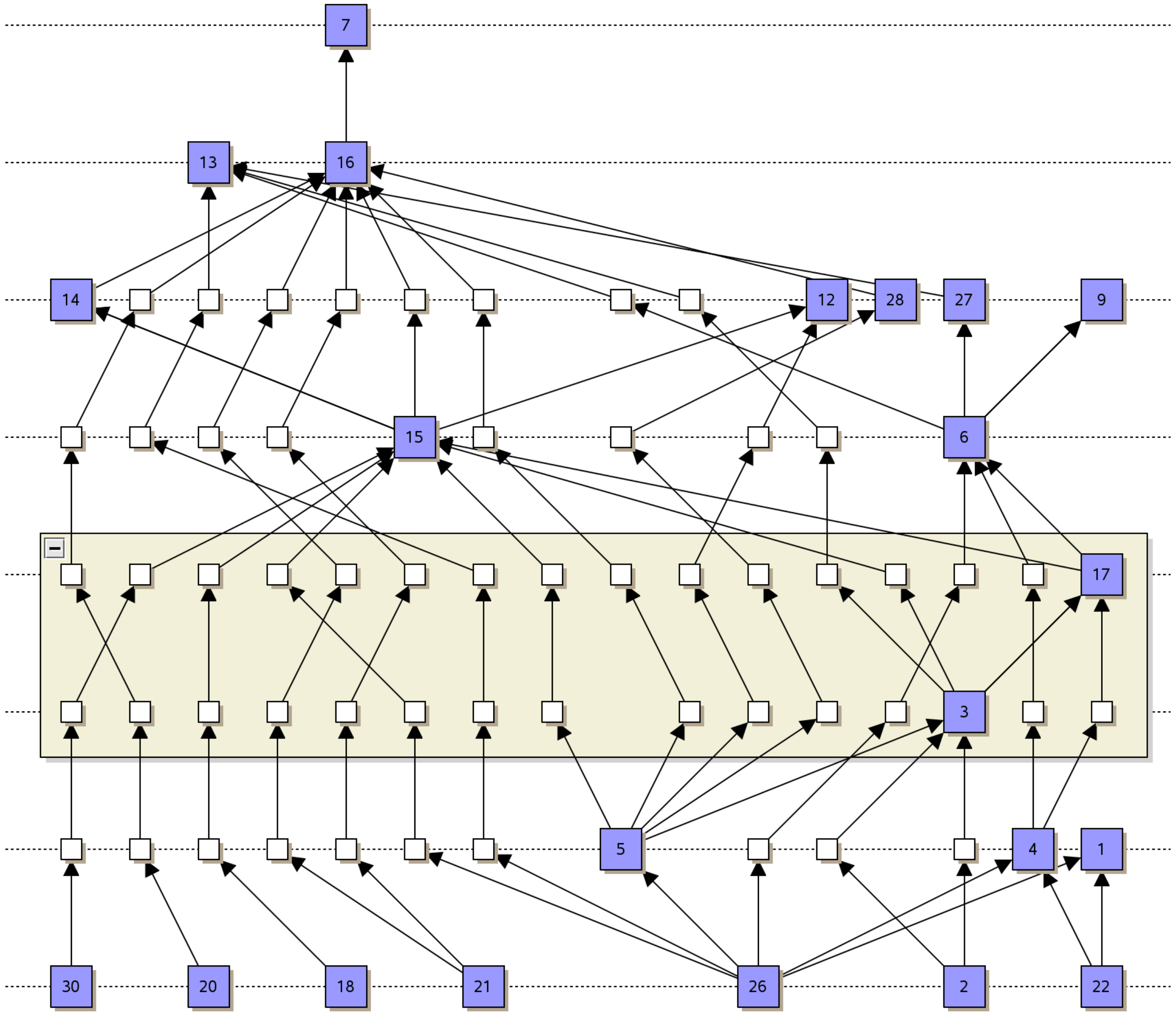
Iterations on example



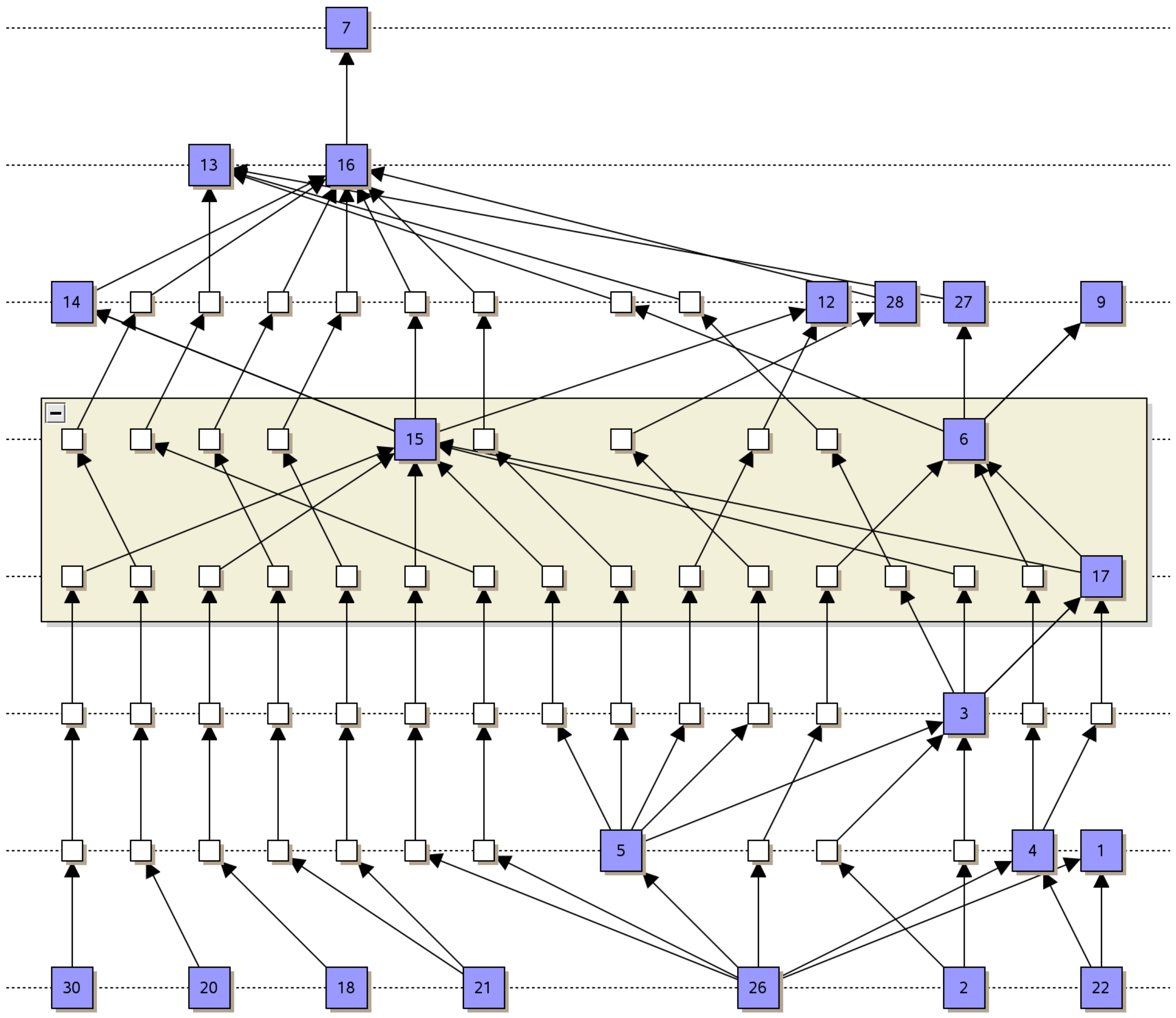
Iterations on example



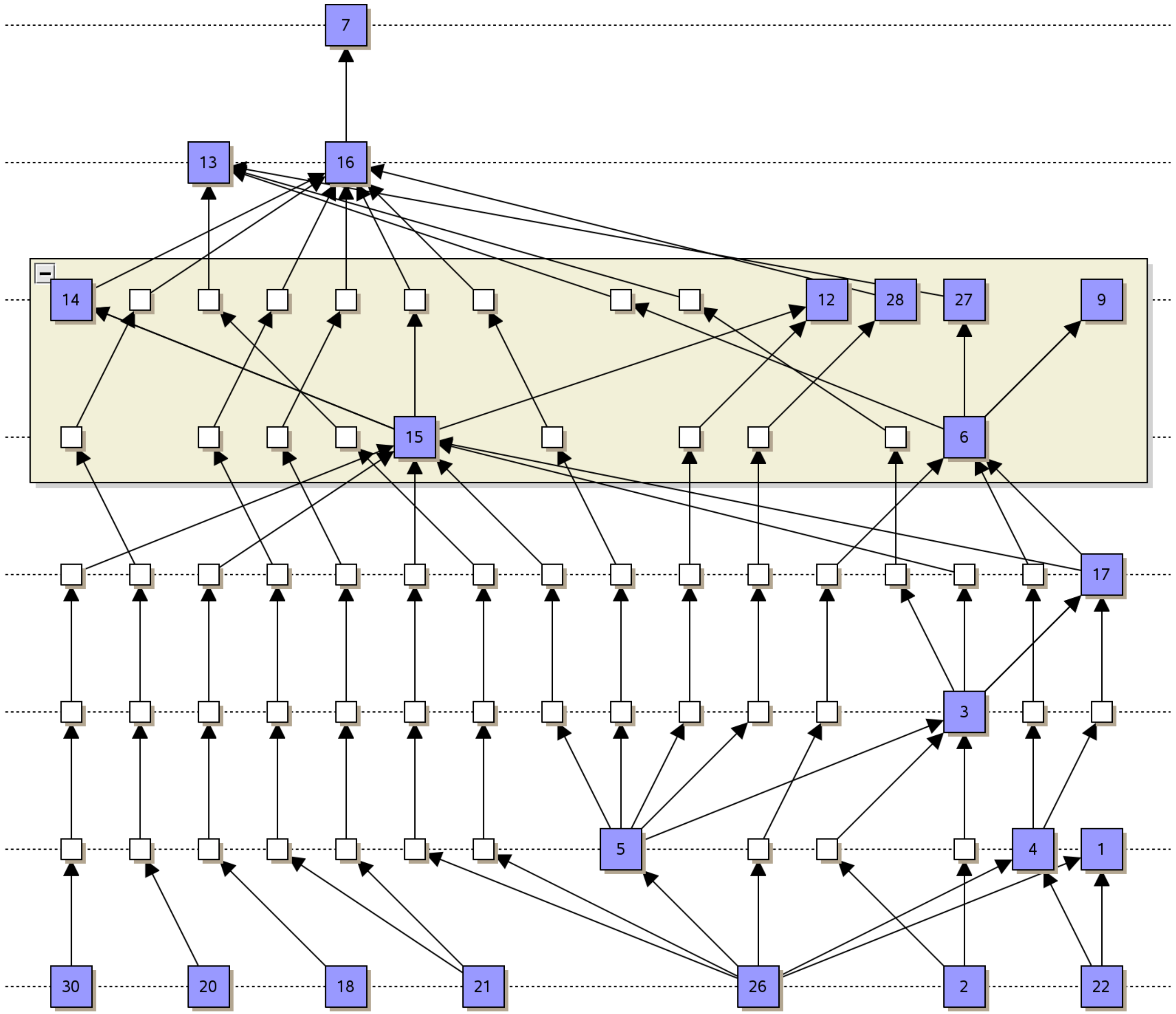
Iterations on example



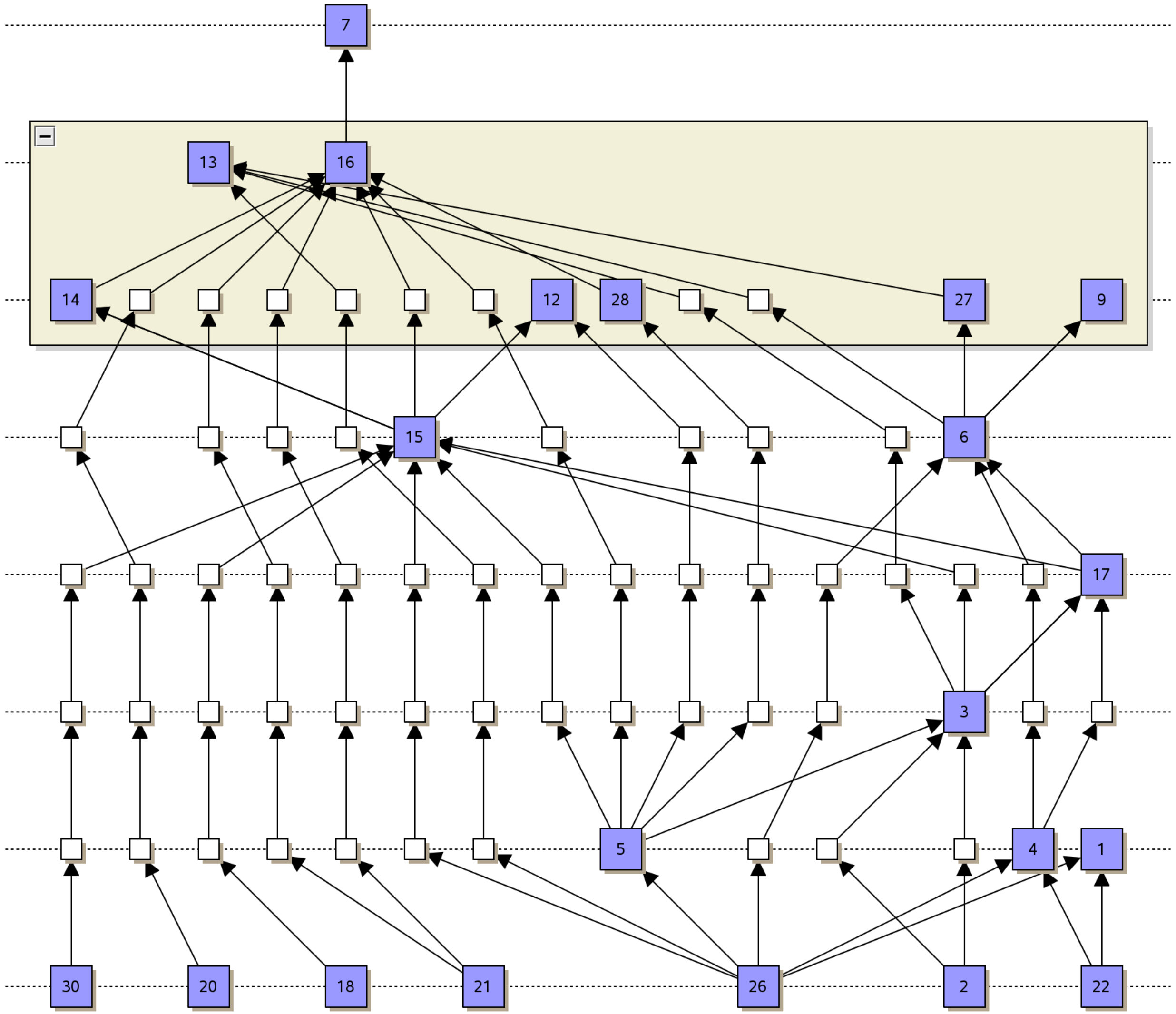
Iterations on example



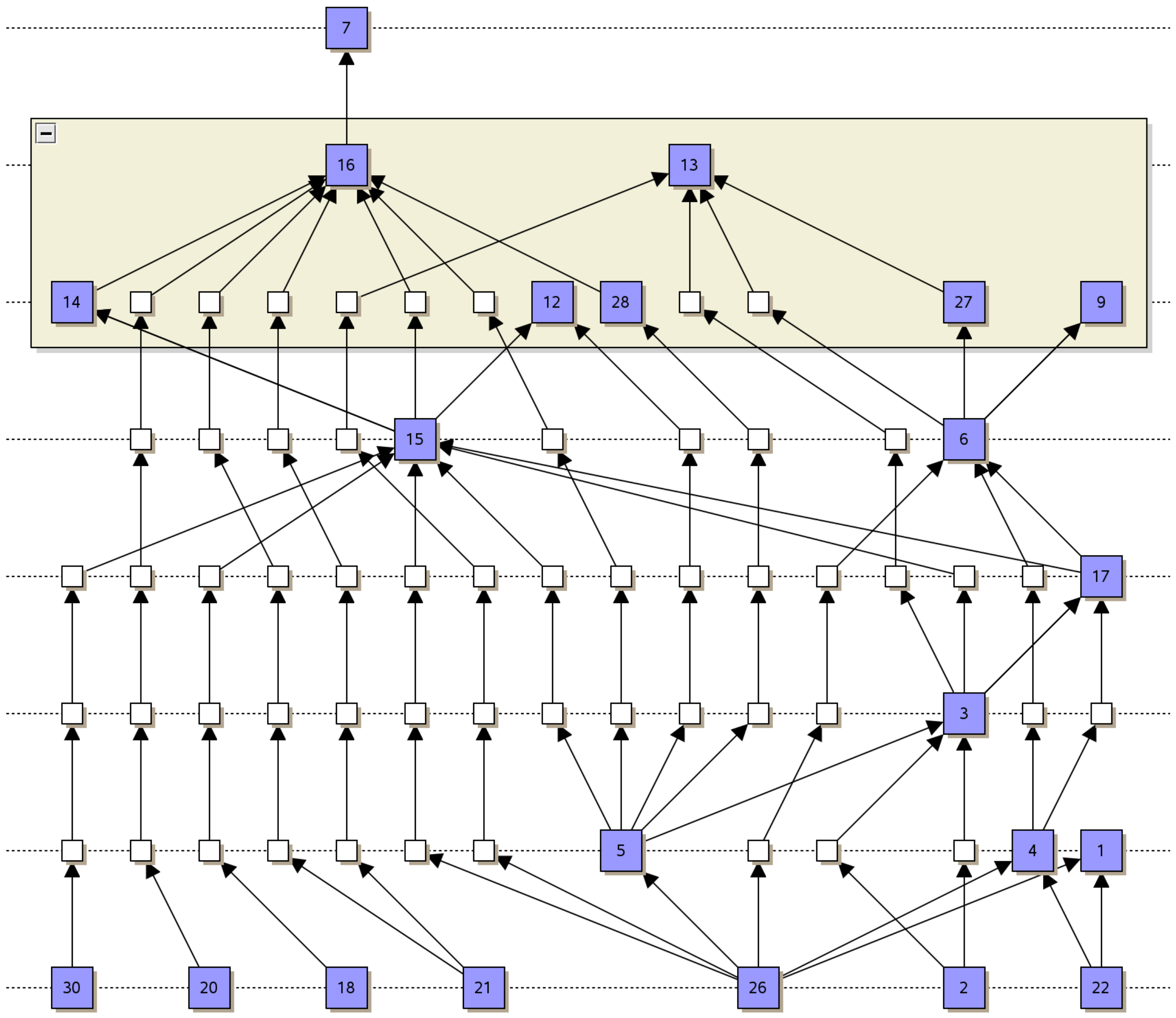
Iterations on example



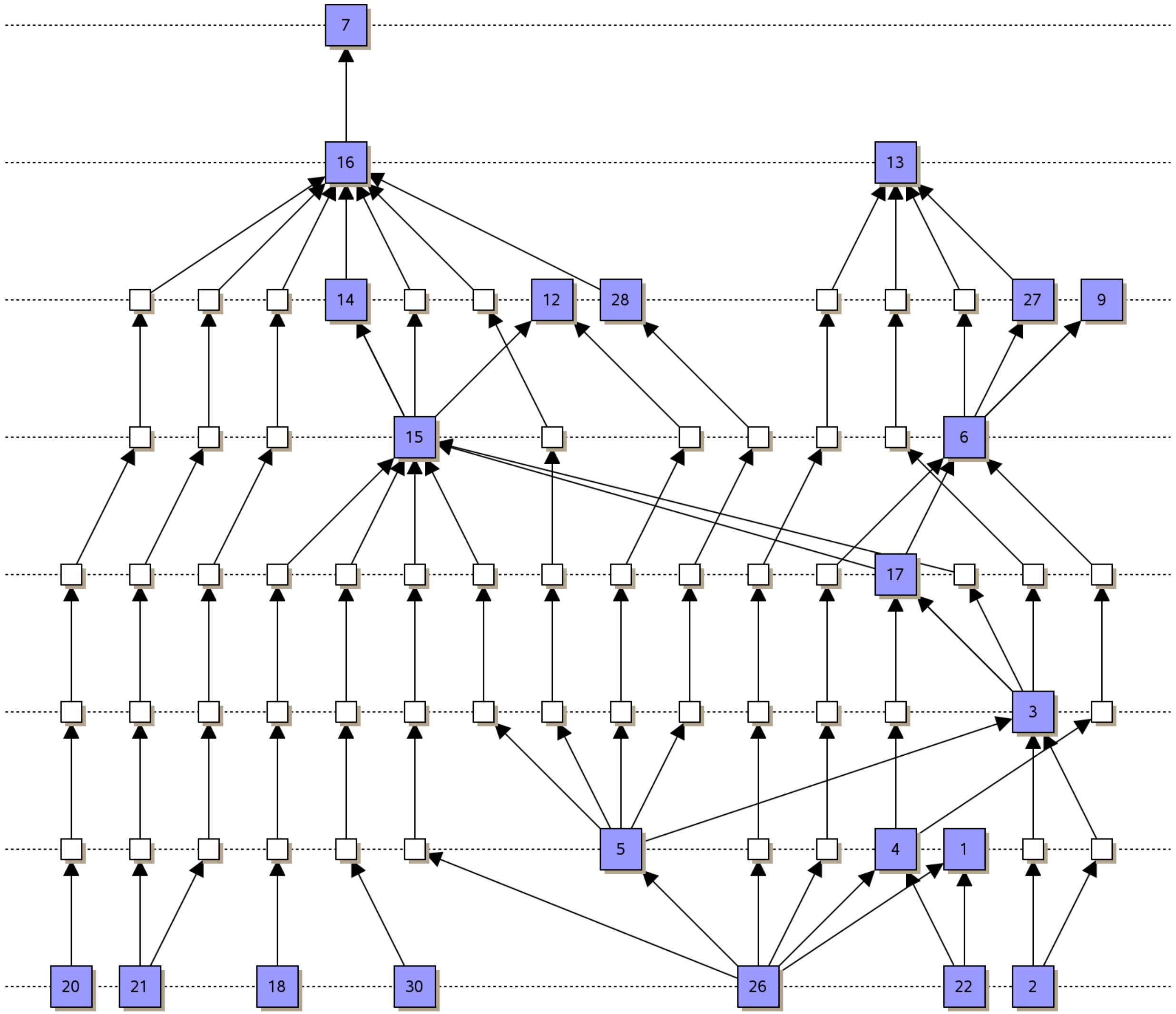
Iterations on example



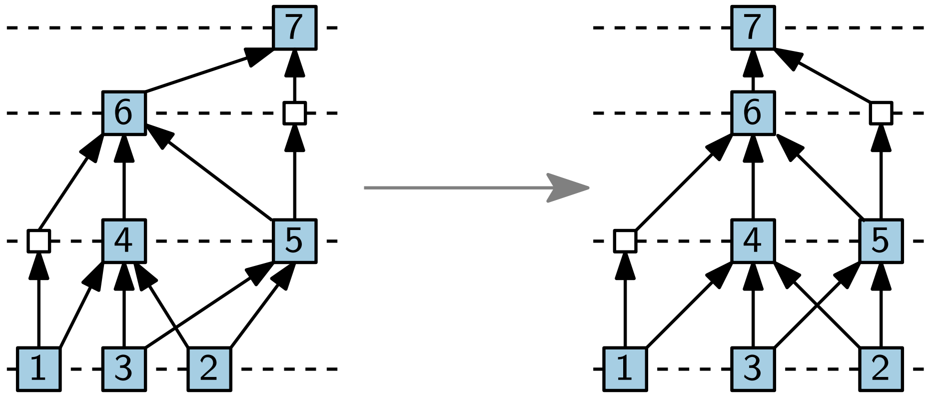
Iterations on example



Iterations on example



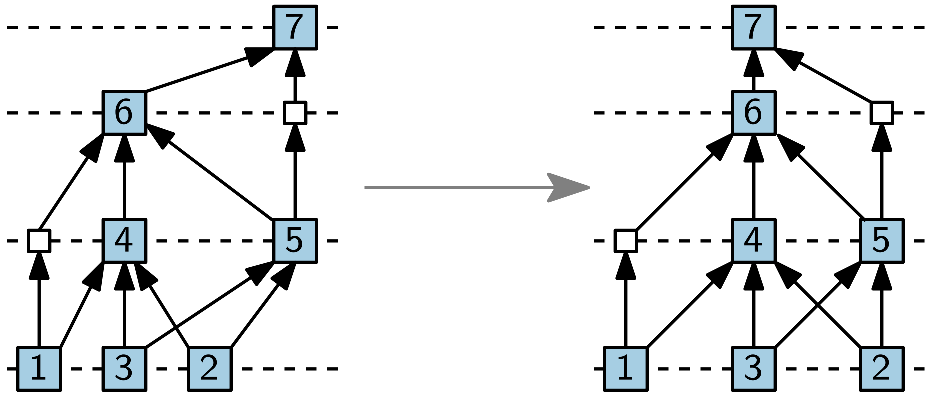
Step 4: Vertex positioning



Goal.

paths should be close to straight, vertices evenly spaced

Step 4: Vertex positioning



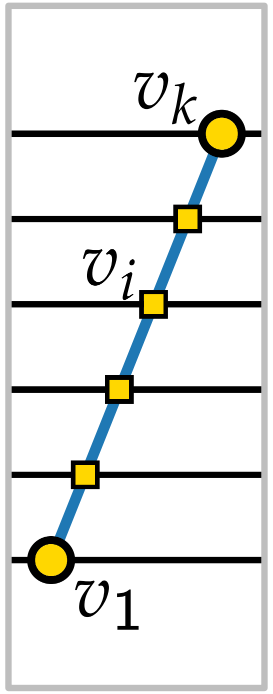
Goal.

paths should be close to straight, vertices evenly spaced

- **Exact:** Quadratic Program (QP)
- **Heuristic:** iterative approach

Quadratic Program

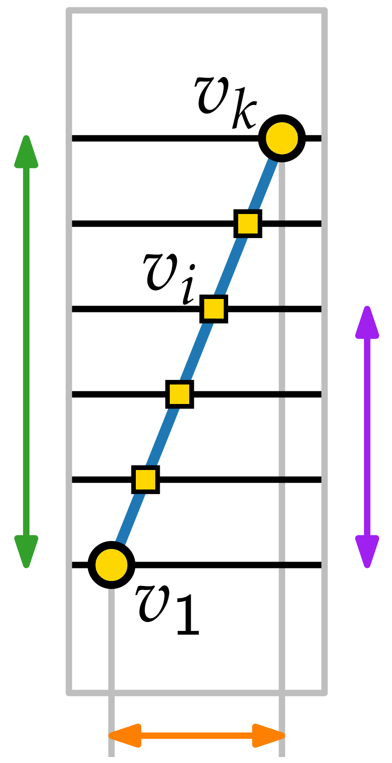
- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$



Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$



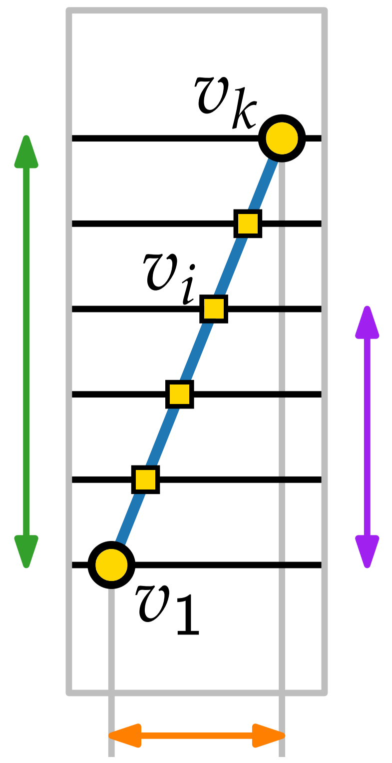
Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$

- define the deviation from the line

$$\operatorname{dev}(p_e) := \sum_{i=2}^{k-1} \left(x(v_i) - \overline{x(v_i)}\right)^2$$



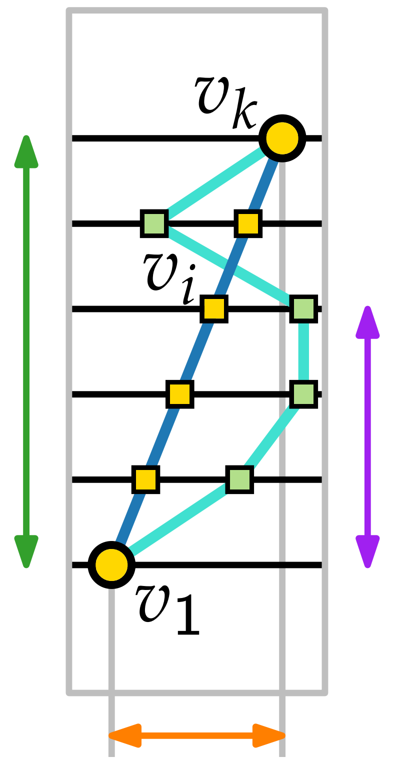
Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$

- define the deviation from the line

$$\mathrm{dev}(p_e) := \sum_{i=2}^{k-1} \left(x(v_i) - \overline{x(v_i)}\right)^2$$



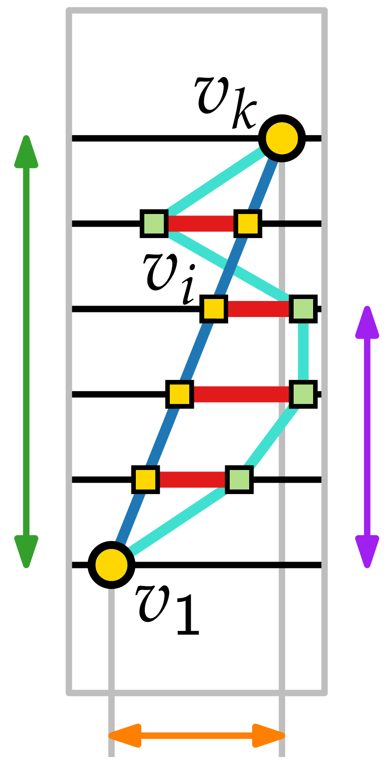
Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$

- define the deviation from the line

$$\operatorname{dev}(p_e) := \sum_{i=2}^{k-1}\left(x(v_i) - \overline{x(v_i)}\right)^2$$



Quadratic Program

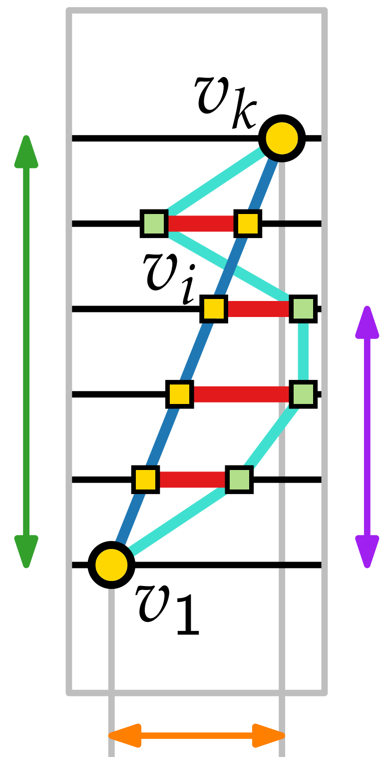
- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$

- define the deviation from the line

$$\text{dev}(p_e) := \sum_{i=2}^{k-1} \left(x(v_i) - \overline{x(v_i)}\right)^2$$

- Objective function: $\min \sum_{e \in E} \text{dev}(p_e)$



Quadratic Program

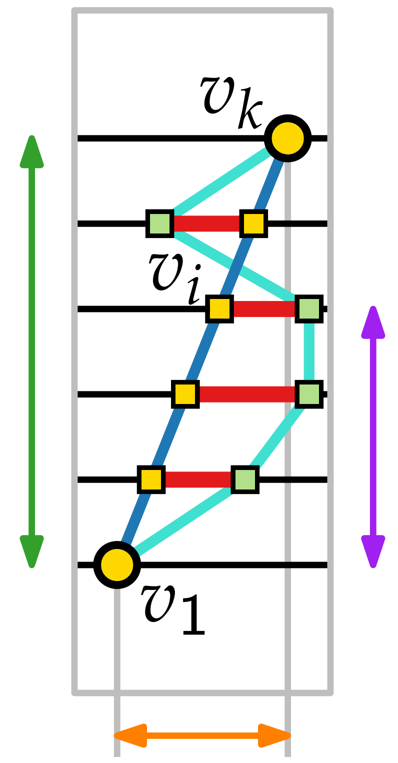
- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$

- define the deviation from the line

$$\mathrm{dev}(p_e) := \sum_{i=2}^{k-1} \left(x(v_i) - \overline{x(v_i)}\right)^2$$

- Objective function: $\min \sum_{e \in E} \mathrm{dev}(p_e)$
- Constraints for all vertices v, w in the same layer with w right of v: $x(w) - x(v) \geq \rho(w, v)$ ← min. horizontal distance



Quadratic Program

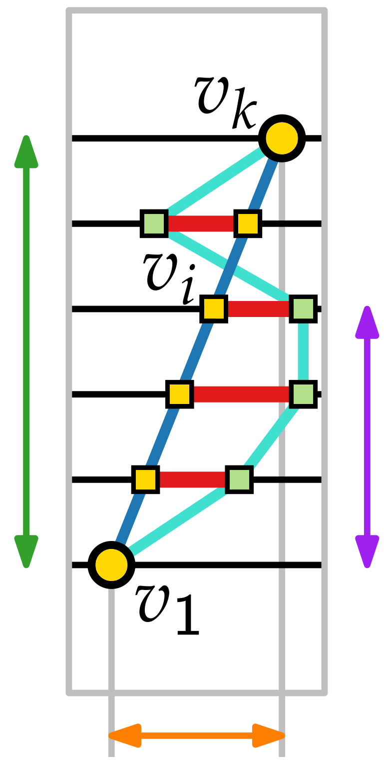
- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$

- define the deviation from the line

$$\mathrm{dev}(p_e) := \sum_{i=2}^{k-1} \left(x(v_i) - \overline{x(v_i)}\right)^2$$

- Objective function: $\min \sum_{e \in E} \mathrm{dev}(p_e)$
- Constraints for all vertices v, w in the same layer with w right of v: $x(w) - x(v) \geq \rho(w, v)$ ← min. horizontal distance



- QP is time-expensive
- width can be exponential

Iterative heuristic

- compute an initial layout

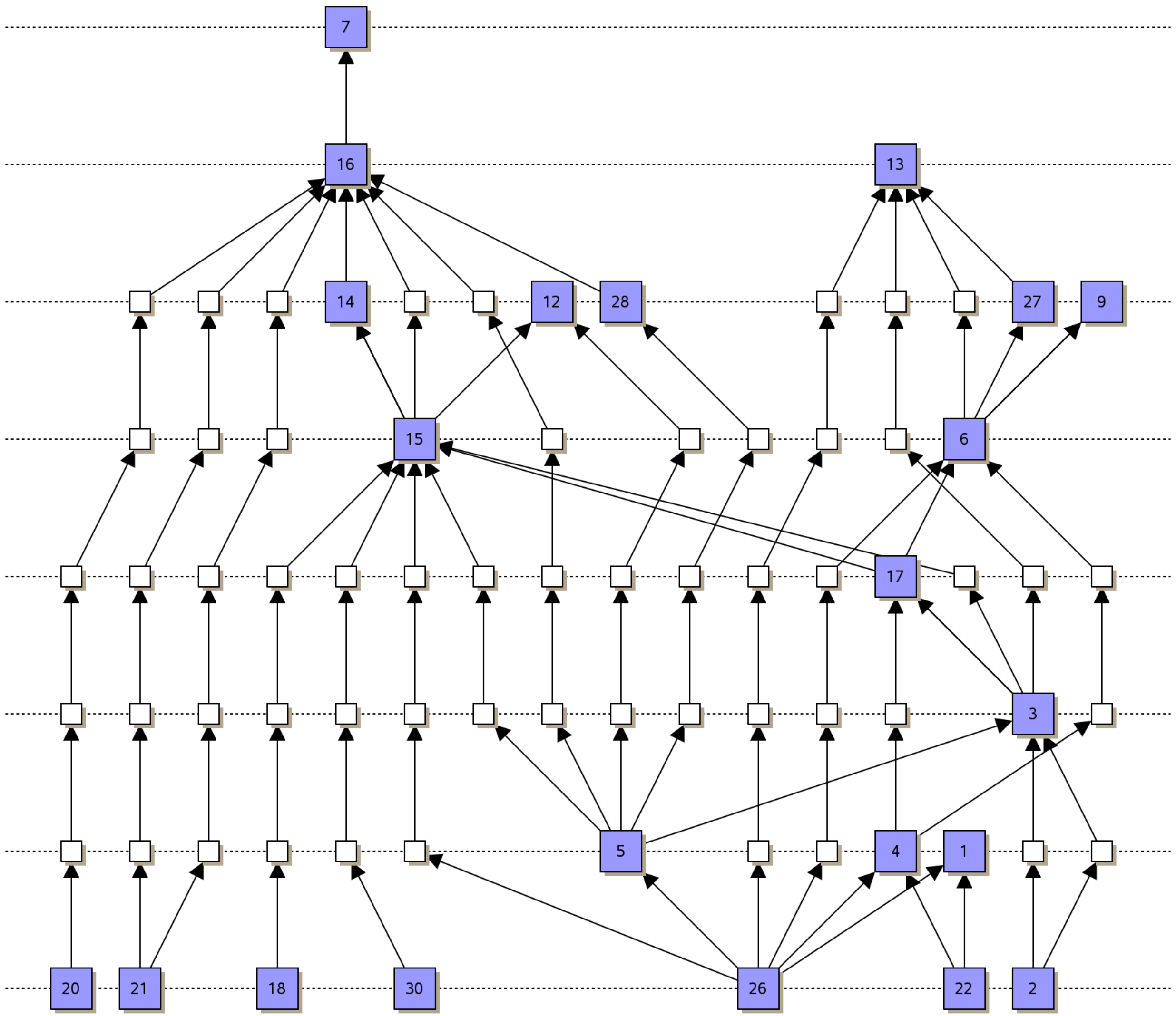
Iterative heuristic

- compute an initial layout
- apply the following steps as long as improvements can be made:

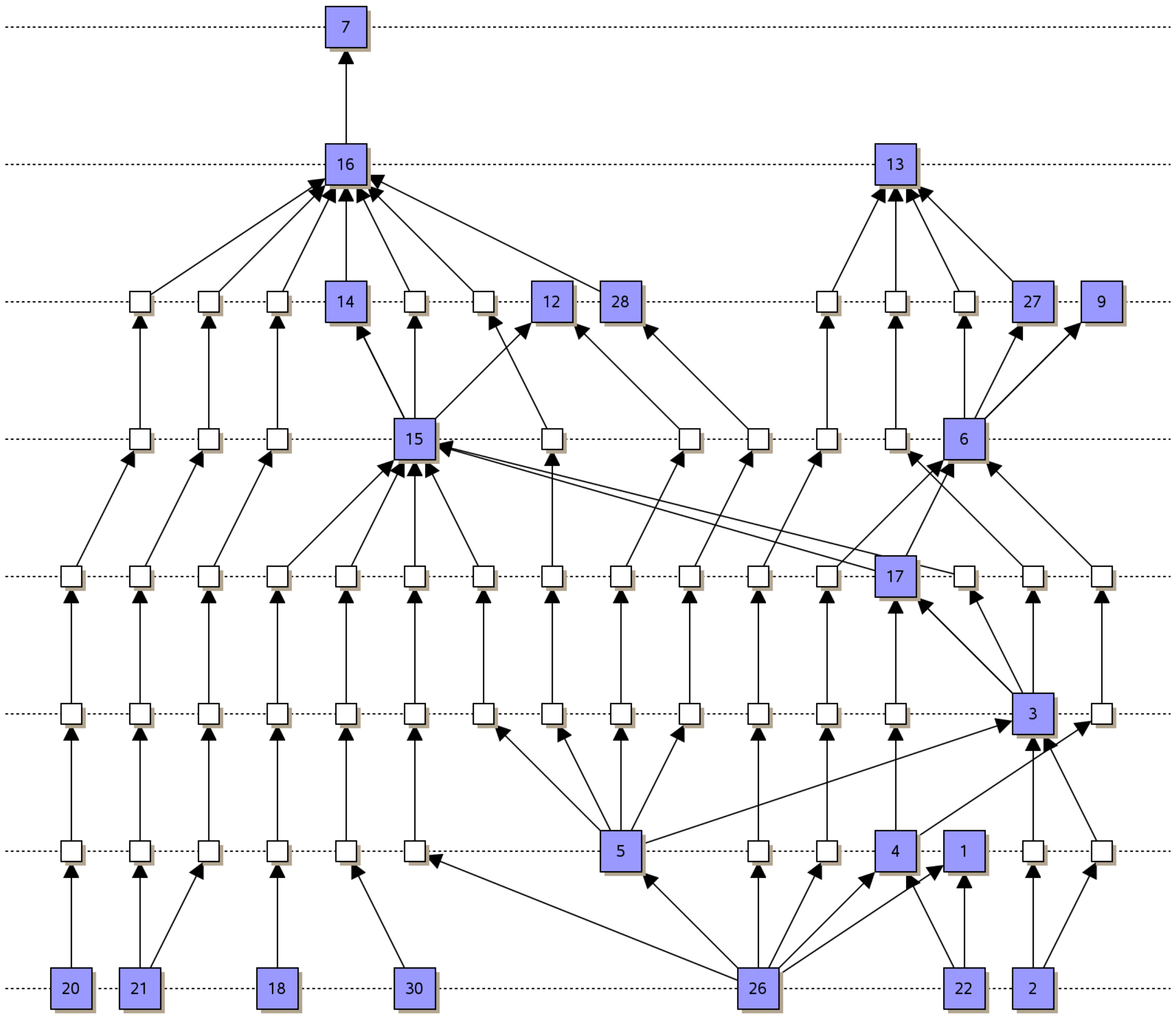
Iterative heuristic

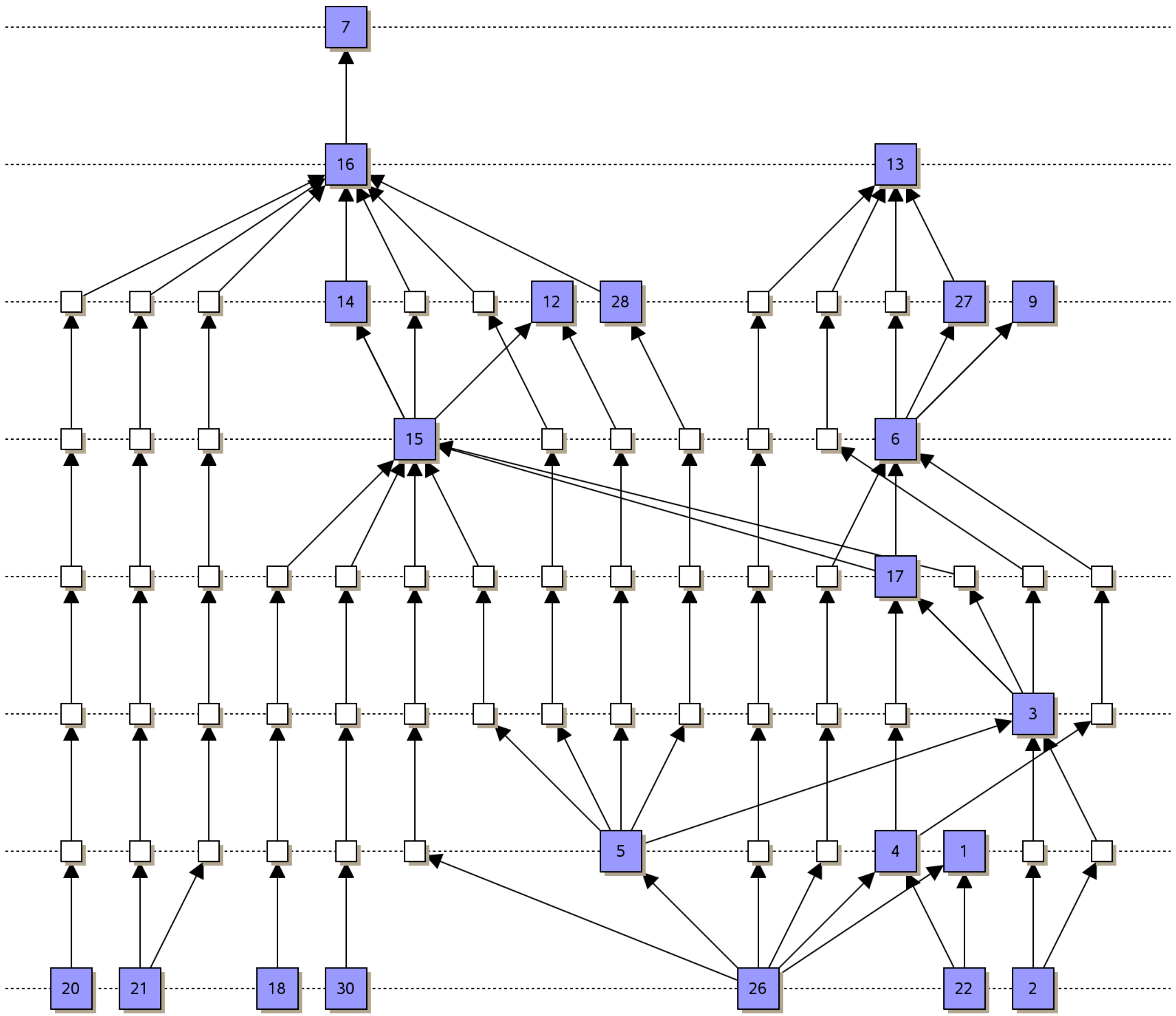
- compute an initial layout
- apply the following steps as long as improvements can be made:
  1. vertex positioning,
  2. edge straightening,
  3. compactifying the layout width.

Example

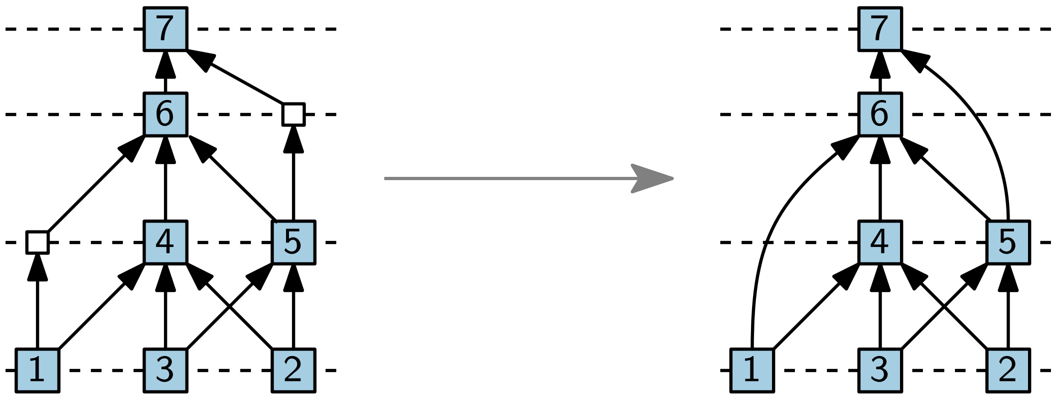


Example


7
16
13
14
12
28
27
9
15
6
17
3
5
4
1
20
21
18
30
26
22
2

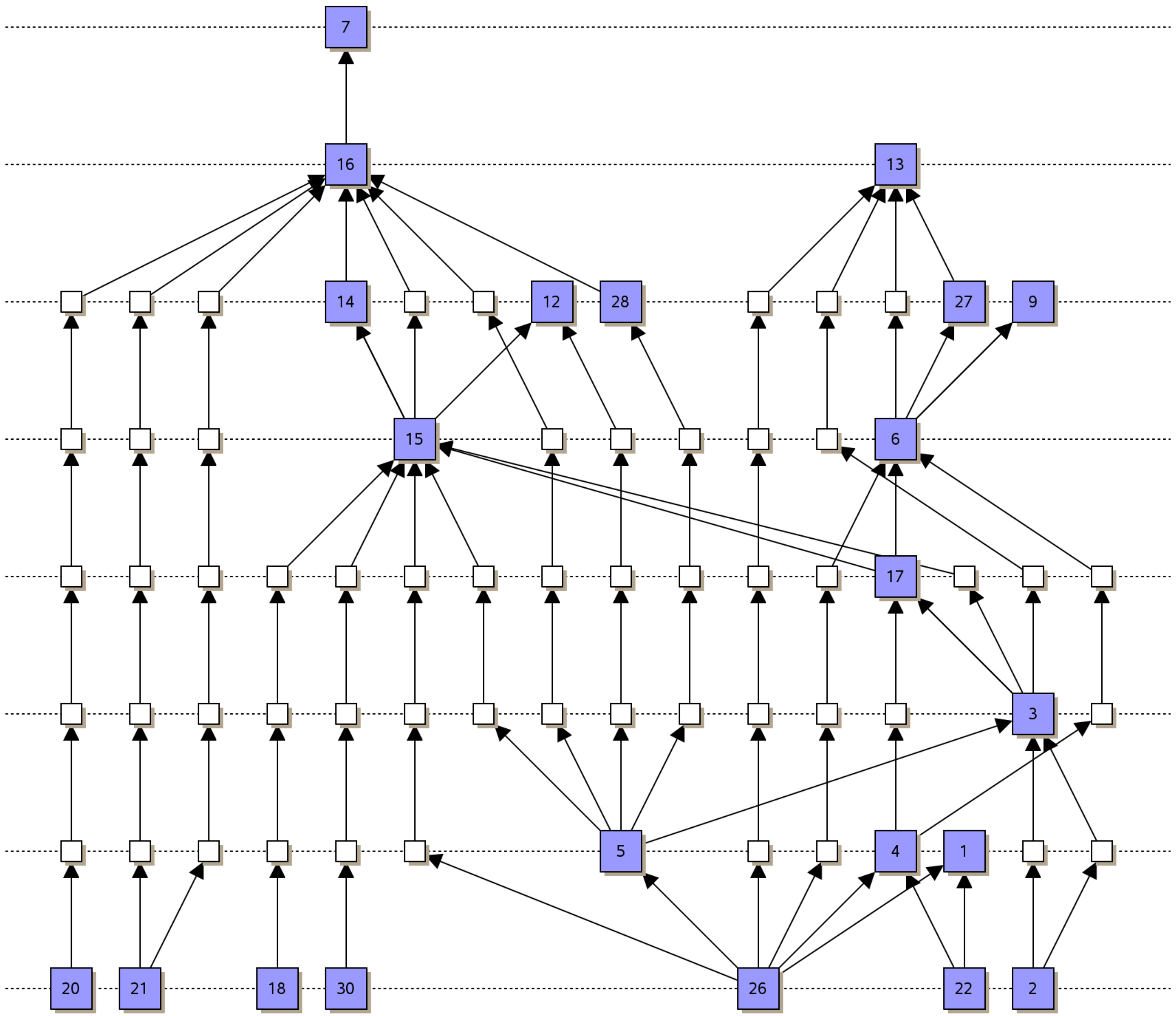

7
16
13
14
12
28
27
9
15
6
17
3
5
4
1
20
21
18
30
26
22
2

Step 5: Drawing edges

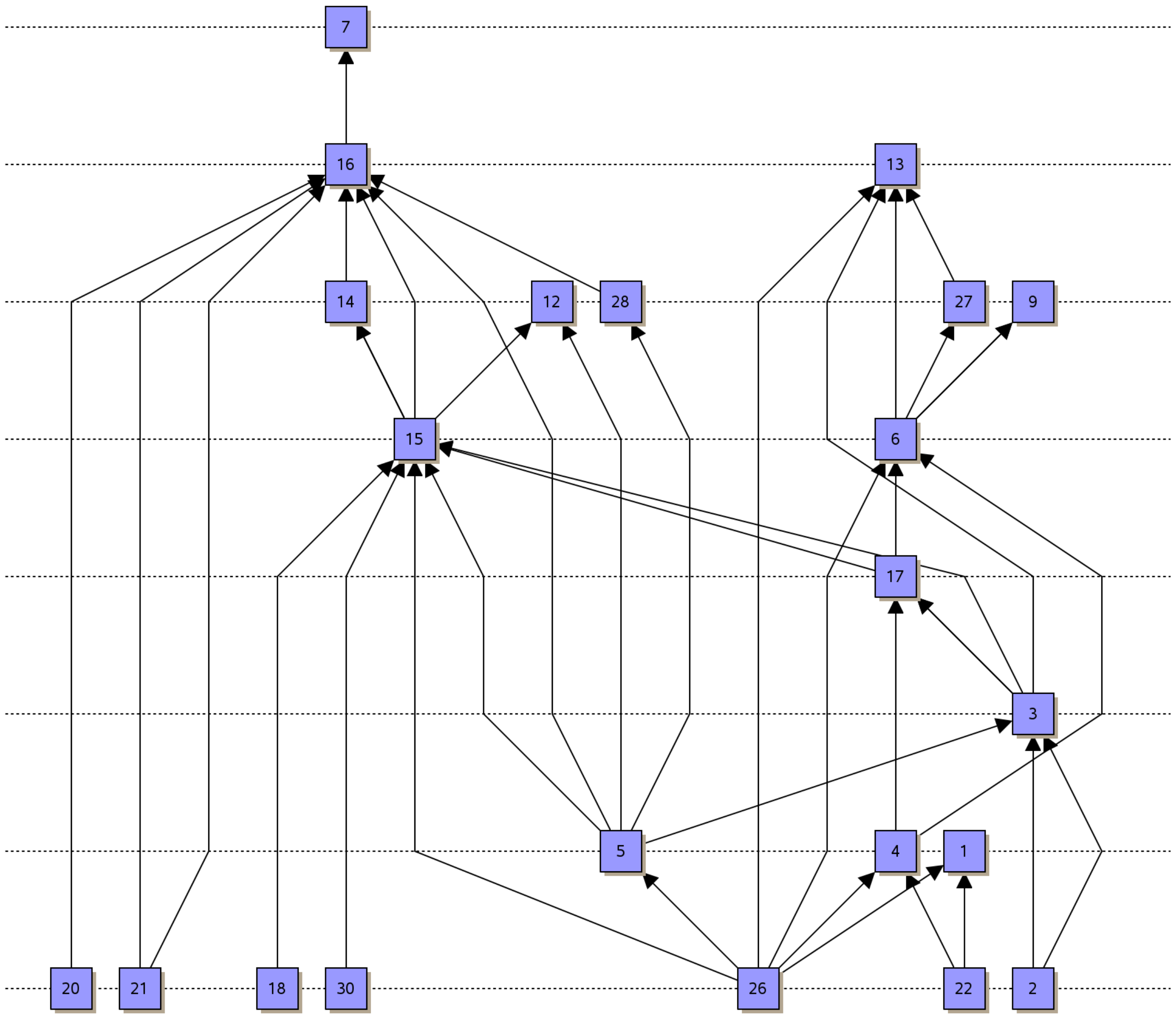


Possibility.
Substitute polylines by Bézier curves

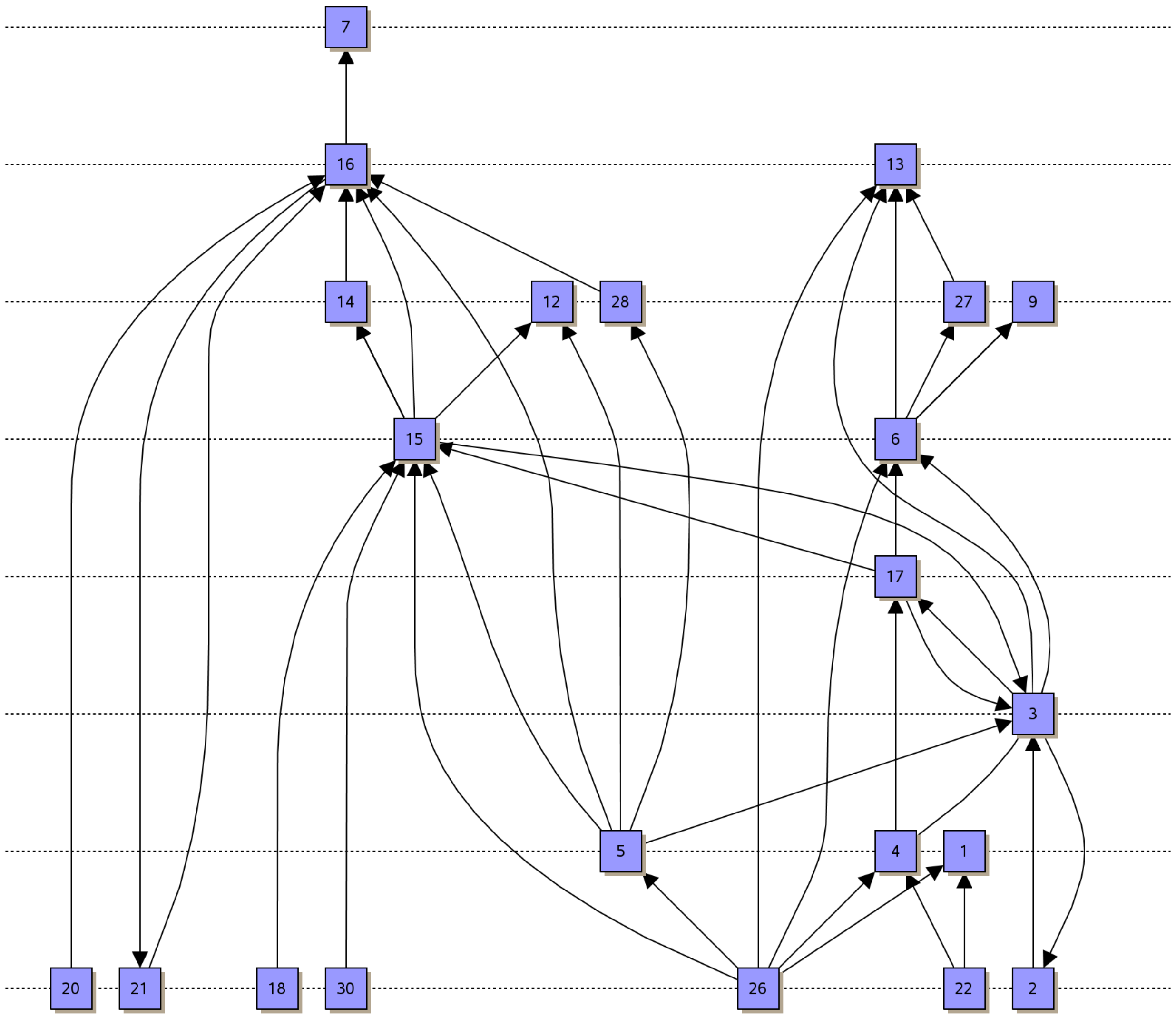
Example



Example

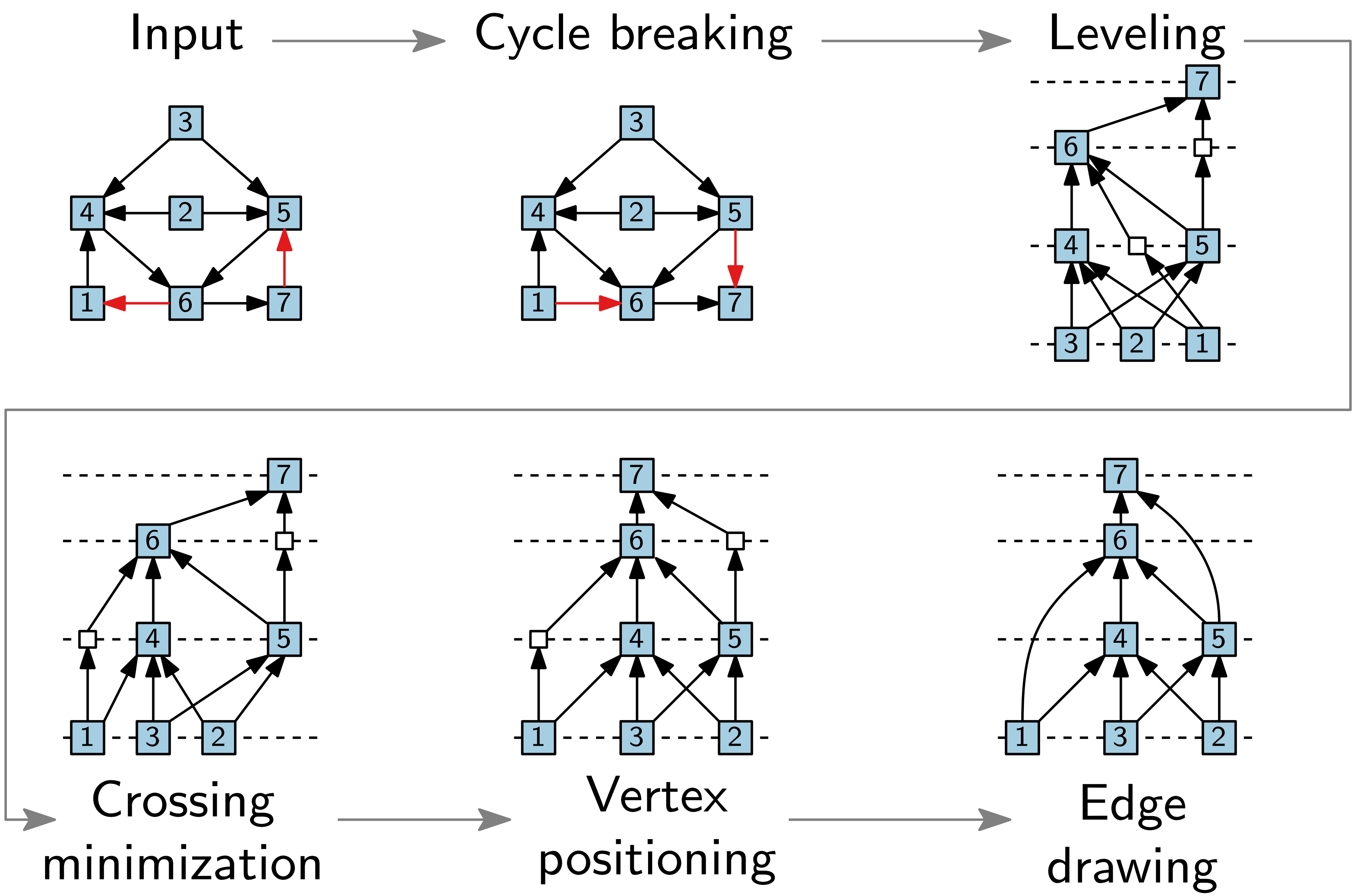


Example



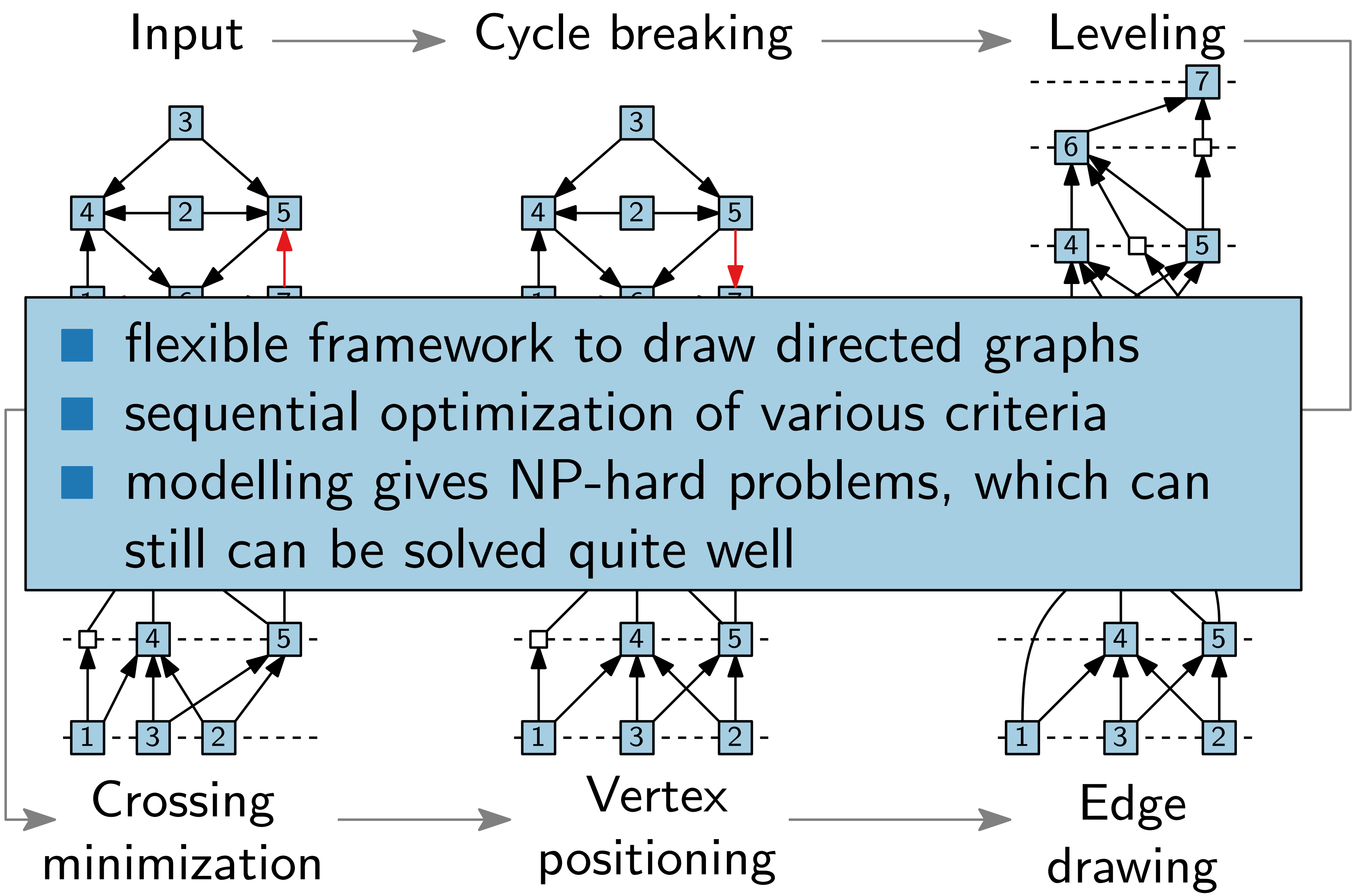
Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]



Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda '81]



- flexible framework to draw directed graphs
- sequential optimization of various criteria
- modelling gives NP-hard problems, which can still can be solved quite well

Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81] Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation alogorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97] 2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms